Published by Institute of Physics Publishing for SISSA

Received: *March 18, 2007*
Accepted: *April 13, 2007*
Published: *May 11, 2007*

1/8 BPS States in AdS/CFT

Edi Gava,[abc] **Giuseppe Milanesi,**[c] **K.S. Narain**[a] **and Martin O'Loughlin**[a]

[a] *High Energy Section, The Abdus Salam International Centre for Theoretical Physics*
Strada Costiera 11, 34014 Trieste, Italy
[b] *Istituto Nazionale di Fisica Nucleare, sez. di Trieste*
Trieste, Italy
[c] *Scuola Internazionale Superiore di Studi Avanzati*
Via Beirut 2-4, 34014 Trieste, Italy
E-mail: gava@ictp.it, milanesi@sissa.it, narain@ictp.it, Martin.John.O.Loughlin@p-ng.si

Abstract: We study a class of exact supersymmetric solutions of type IIB Supergravity. They have an $SO(4) \times SU(2) \times U(1)$ isometry and preserve generically 4 of the 32 supersymmetries of the theory. Asymptotically $AdS_5 \times S^5$ solutions in this class are dual to 1/8 BPS chiral operators which preserve the same symmetries in the $\mathcal{N} = 4$ SYM theory. They are parametrized by a set of four functions that satisfy certain differential equations. We analyze the solutions to these equations in a large radius asymptotic expansion: they carry charges with respect to two $U(1)$ KK gauge fields and their mass saturates the expected BPS bound.

Keywords: AdS-CFT Correspondence, Global Symmetries.

Contents

JHEP05(2007)030

1. Introduction

One of the first steps in understanding the AdS/CFT correspondence is to set up a precise dictionary between the states of the theories on the two sides of the correspondence. It is well known that the parameters of the $\mathcal{N} = 4$ $SU(N)$ SYM, namely $\lambda \equiv g_{YM}^2 N$ and N should be identified with the parameters of type IIB String Theory on $AdS_5 \times S^5$, namely L_{AdS}, ℓ_s, g_s, as

$$\frac{L_{AdS}^2}{4\pi\ell_s^2} = \left(\frac{\lambda}{4\pi}\right)^{\frac{1}{2}} \qquad g_s = \frac{\lambda}{N}\,. \tag{1.1}$$

Type IIB Supergravity is a good approximation of String Theory at low energies compared to the string scale and small string coupling. We may thus consider solutions to the supergravity equations of motion which are asymptotically $AdS_5 \times S^5$ as good candidates

for dual of states in the CFT, provided that $N \gg \lambda \gg 1$ and any dimension four curvature invariant of the solutions, $\mathcal{R}_4$, satisfies

$$\mathcal{R}_4 L_{AdS}^4 \ll \lambda \ll N\,. \tag{1.2}$$

One may hope to be able to carry out this program in the full BPS sector of the respective dual theories. A very beautiful and relatively simple construction of such a dictionary in the half BPS sector has been performed in [1](LLM). The authors considered geometries dual to half BPS states in the CFT, associated to chiral primary operators which are obtained by taking traces of powers of the operator $Z \equiv Z_3 \equiv \phi_5 + \mathrm{i}\phi_6$, where $\{\phi_i\}_{1\leq i\leq 6}$ are the six adjoint scalars of the $\mathcal{N} = 4$ $SU(N)$ gauge theory. In LLM, exact half BPS solutions to the supergravity equations of motion are derived by exploiting the $\mathbb{R} \times SO(4) \times SO(4)$ bosonic symmetry of the problem. The complete geometry, together with the self-dual 5-form, are determined by a single function z which is defined on a three dimensional halfspace and which satisfies a linear elliptic differential equation: solutions are thus specified by boundary conditions at infinity and on the boundary plane. LLM were able to identify the boundary conditions giving rise to non-singular asymptotically $AdS_5 \times S^5$ geometries. The resulting space of classical solutions can be directly identified with the phase space of the dual states of the gauge theory in the free fermion picture [2, 3]. The latter emerges after reducing the (single) scalar sector of the gauge theory on $\mathbb{R} \times S^3$. These solutions represent the geometrical transition between probe giant gravitons or dual giant gravitons [4 – 7] and fully backreacted geometries. A giant graviton is a classical D3-brane configuration wrapping an $S^3 \subset S^5$ and rotating along an equator of the S^5. A dual-giant graviton is another half-BPS D3-brane configuration that wraps an $S^3 \subset AdS_5$.

It is natural to ask how the above very precise correspondence between geometry on the one hand, and features of the quantum mechanical states of the reduced gauge theory on the other, extends to cases with less supersymmetry. In the recent literature there have been various attempts in this direction: for example, in [8] one quarter BPS geometries were found by assuming a non trivial axion-dilaton. This corresponds to putting smeared D7 branes in the background and thus to adding flavour to the gauge theory. A description of one eighth and one quarter BPS geometries in the language of five dimensional gauged supergravity has been given in [9]. The construction of a class of one quarter BPS solutions directly in type IIB appeared in [10, 11]. Another interesting related work is presented in [12]. This problem was also approached in the probe approximation, where the backreaction on the geometry is neglected: D3 branes can wrap more complex three dimensional surfaces in S^5 and give rise to giant gravitons with fewer supersymmetries [13]. In [14] the authors have been able to count such states. The quantization of their classical phase space has been performed in [15]. Other works that present interestng connection with ours can be found in [16 – 18].

In this paper we address the problem of finding BPS supergravity solutions which represent the fully backreacted geometry of a class of 1/8 BPS giant gravitons. Our solutions correspond to gauge theory states associated to linear combinations of composite operators

$$\mathcal{O}(q, r) = \mathrm{Tr}(Z_1^q)\mathrm{Tr}(Z_2^q)\mathrm{Tr}(Z_3^r) + \cdots\,. \tag{1.3}$$

where Z_1, Z_2 and Z_3 are the three complex scalars of the $\mathcal{N} = 4$ CFT. The dots signify other terms with suitable (anti)-symmetrization and trace structures, which have all a total of q Z_1 and Z_2 fields and r Z_3 fields. They are chosen such that $\mathcal{O}(q, r)$ are chiral primary operators which are invariant under the $SU(2)_L$ subgroup of the $SU(2)_L \times SU(2)_R$ acting on Z_1, Z_2. We consider linear combinations of $\mathcal{O}(q, r)$ which have all the same value of q but may have different values of r.
The lowest mode $\mathcal{O}(q, r)$ in the expansion on spherical harmonics on S^3 saturates the BPS bound:

$$\Delta = 2q + r\,, \tag{1.4}$$

where Δ is the conformal dimension of the operator. The total amount of bosonic symmetry preserved by the corresponding states is thus given by $SO(4)_{KK} \times SU(2)_L \times U(1)_R$. Consequently, we start from an Ansatz for the metric and the self-dual RR 5-form which preserves this amount of symmetry. This implies, as for LLM, that the resulting background will depend non-trivially on three coordinates (an additional symmetry will be associated to the time coordinate, like in LLM). We also require that the background possesses the required amount of supersymmetry by demanding that it possesses a Killing spinor. Applying techniques similar to those in [1, 19 – 23] we have been able to express the full solution in terms of four independent functions defined on a three dimensional half-space. As a result of certain Bianchi identities and integrability conditions, these four functions have to satisfy a system of nonlinear, coupled, elliptic differential equations. A unique solution to these equations is obtained once a set of boundary conditions at infinity and on the boundary plane is specified; boundary conditions should be chosen in such a way as to give non-singular geometries with $AdS_5 \times S^5$ asymptotics.

We present here the boundary conditions that give rise to asymptotically non-singular $AdS_5 \times S^5$ geometries. We solve the equations asymptotically up to third order in a large radius expansion. From this analysis we can extract the two dimensionless charges Q and J carried by the solution. These are the charges corresponding to two out of the three $U(1)$ Cartan gauge fields arising from the KK reduction of IIB supergravity on S^5 to five dimensional maximal gauged supergravity. These charges in turn correspond to the q and r charges of the gauge theory side. Moreover, we verify that our solutions saturate the expected BPS bound:

$$M = \frac{\pi L_{AdS}^2}{4G_5}(|J| + 2|Q|)\,. \tag{1.5}$$

Unfortunately, a more exhaustive analysis of such boundary conditions is quite difficult due to the complexity (non linearity) of the differential equations. In other words we do not know which of the boundary conditions give rise to globally non-singular backgrounds. We will comment on this issue in the conclusions. The paper is organized as follows. In Section 2 we present the gauge theory description of the 1/8 BPS states that we wish to study. In Section 3 we show how the 1/8 supersymmetry constrains the components of the metric and 5-form and we reduce these constraints to four differential equations on four scalar functions. In Section 4 we present the large radius asymptotic analysis. Appendix A sets our conventions. In appendix B the complete derivation of the results presented in

JHEP05(2007)030

section 3 is given and in appendix C we make some observations on the formal tools used to facilitate the analysis. Due to the complexity of the equations involved, we performed the complete analysis by means of the software Mathematica. All the derivations that are not described in full detail in the text were obtained with the help of such software.

2. Gauge theory analysis

Chiral primary operators of the $\mathcal{N}=4$ superconformal algebra $SU(2,2|4)$ were classified in [24 – 26] and are characterized by the number of Poincare' supersymmetries they preserve. They can preserve $\frac{1}{2}$, $\frac{1}{4}$ or $\frac{1}{8}$ of the 16 Poincare' supersymmetries. We will be interested in operators that are composites of the six adjoint scalars ϕ_i of the $\mathcal{N}=4$ $SU(N)$ gauge theory. The simplest class is that of $\frac{1}{2}$-BPS operators the elements of which are characterized by the $SO(6)$ R-symmetry representations given by Young tableaux with a single row of length p, i.e. traceless, symmetric $SO(6)$ rank p tensors, or in Dynkin notation, the $[0,p,0]$ of $SU(4)$. In this case the conformal dimension Δ is equal to p. The highest weight of this representation can be obtained by using one of the complex adjoint scalars, $Z \equiv Z_3 \equiv \phi_5 + i\phi_6$, which has charge 1 with respect to the $SO(2)$ generator, $J_3 \equiv L_{5,6}$[1] , in $SO(6)$. One can construct from Z the multitrace composite $SU(N)$ singlet operators with $\Delta = p$. These operators therefore preserve an $SO(4) \subset SO(6)$ times an $SO(4) \subset SO(4,2)$, since the only modes satisfying the relation $\Delta = p$ are the S^3 scalars. This $SO(4) \times SO(4)$ symmetry has been used in LLM as the isometry of the supergravity background and it is the key for their (relatively) simple and beautiful solution. The lower supersymmetry cases are again best described in terms of $SO(6)$ Young tableaux: the $\frac{1}{4}$ and $\frac{1}{8}$ cases correspond to tableaux with two rows (of lengths p, q, $p \geq q$) and three rows (of lengths p, q, r, $p \geq q \geq r$) respectively. The conformal dimensions saturate the bounds $p+q$ and $p+q+r$ respectively. Again, in discussing highest weight states it is convenient to use the three complex scalars $Z_1 = \phi_1 + i\phi_2$, $Z_2 = \phi_3 + i\phi_4$ and Z_3, which have charges (1,0,0), (0,1,0) and (0,0,1) with respect to the three Cartan generators (J_1, J_2, J_3) $SU(4)$ respectively. Highest weight states saturate the BPS bound $\Delta = J_1 + J_2 = p + q$ and $\Delta = J_1 + J_2 + J_3 = p + q + r$ in the $\frac{1}{4}$ and $\frac{1}{8}$ case, respectively.

This is summarized in the following table:

$p = q = 0,\ r \neq 0$	$p, q \neq 0,\ r = 0$	$p, q, r \neq 0$
$1/2\,BPS$	$1/4\,BPS$	$1/8\,BPS$

Let us consider the $\frac{1}{8}$ case: given the three complex scalars Z_1, Z_2, Z_3 of the $\mathcal{N}=4$ $SU(N)$ Super Yang Mills theory one can construct a basis of gauge invariant, local, composite operators in the $[p,q,r]$ of the R-symmetry group $SU(4)$ as [27]

$$\mathrm{Tr}(Z_1^p)\mathrm{Tr}(Z_2^q)\mathrm{Tr}(Z_3^r) + \cdots . \tag{2.1}$$

where the dots mean suitable (anti)-symmetrization and trace structure that projects to the chiral primaries in the (p,q,r) representation of $SU(4)$.

[1]Here $J_i = L_{2i-1,2i}$ in terms of the standard generators of $SO(6)$

We are interested in constructing duals of the states corresponding to such operators. However, generic operators of this type break fully the non-abelian $SO(6)$ R-symmetry, up to possible $U(1)$ factors which act by an overall phase on them. However, if

$$p = q\,. \tag{2.2}$$

we can construct operators which are invariant under the $SU(2)_L$ of the $SO(4) = SU(2)_L \times SU(2)_R$ which rotates the four real scalars

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} \Re\mathrm{e}\, Z_1 \\ \Im\mathrm{m}\, Z_1 \\ \Re\mathrm{e}\, Z_2 \\ \Im\mathrm{m}\, Z_2 \end{pmatrix}\,. \tag{2.3}$$

This is best seen by observing that $SU(2)_L$ and $SU(2)_R$ act as left and right multiplication respectively on:

$$\begin{pmatrix} Z_1 & -\bar{Z}_2 \\ Z_2 & \bar{Z}_1 \end{pmatrix}$$

Therefore Z_1 and Z_2 transform as a doublet of $SU(2)_L$, whereas they have the same charge under $J_R^3 = \frac{J_1+J_2}{2}$. The operators with $p = q$ are clearly singlets of $SU(2)_L$, and they acquire an overall phase under J_R^3. They satisfy the relation

$$\Delta = 2q + r\,. \tag{2.4}$$

The bosonic symmetry preserved by these states is:

$$\mathbb{R}_{BPS} \times \big(SU(2)_L \times U(1)_R\big)_{R-\text{charge}} \times SO(4)_{KK} \tag{2.5}$$

where the first $\mathbb{R}$ corresponds to the transformations generated by

$$D' \equiv D - 2J_R^3 - J\,, \tag{2.6}$$

with $J = J_3$ acting on Z, D is the dilatation operator and the last $SO(4)$ factor represents the fact that we are considering s-wave modes on S^3 in the reduction of SYM theory on $\mathbb{R} \times S^3$ [28]. These are the symmetries that will motivate the Ansatz for the metric and five-form on the supergravity side: we will keep a round 3-sphere with the $SO(4)$ isometry corresponding to the $SO(4)$ above. Another S^3 (related to the $SO(4)$ R-symmetry of the $\frac{1}{2}$ BPS case) which is in the S^5 of the $AdS_5 \times S^5$ background, will be squashed with isometry group reduced to $SU(2)_L \times U(1)_R$.

It will be useful for the subsequent analysis of the Killing spinor equation on the supergravity side, to understand the quantum numbers of the preserved supersymmetries. In an $\mathcal{N} = 1$ and $SU(3) \times U(1) \subset SU(4)$ notation, the supersymmetry variations of the complex scalars Z_i are:

$$\delta Z_i = \xi_i \lambda + \xi \psi_i + \epsilon_{ijk} \bar{\xi}^j \bar{\psi}^k, \tag{2.7}$$

Here the two-component spinors λ and ψ_i are the gaugino and the chiral matter fermions, while ξ and ξ_i are the supersymmetry parameters. They are in the $\mathbf{1}_{3/2}$ and $\mathbf{3}_{-1/2}$ of

$SU(3)_{U(1)}$ respectively. More precisely the Cartan charges of λ, ξ are $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and those of ψ_1, ξ_1 are $(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, and similarly for $\psi_{2,3}, \xi_{2,3}$. From (2.7) it is clear that the highest weight $\frac{1}{8}$ BPS operators are invariant under the supersymmetry corresponding to $\bar{\xi}$. As for the $SU(2)_L \times SU(2)_R = SO(4) \subset SU(3)$ quantum numbers, the roots of $SU(2)_L$ are $(\pm 1, \mp 1, 0)$ and those of $SU(2)_R$ are $(\pm 1, \pm 1, 0)$. Therefore the preserved supersymmetry parameter $\bar{\xi}$, whose charges are $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, is a singlet of the unbroken $SU(2)_L$ and lowest weight with respect to the broken $SU(2)_R$.

3. Generic solutions

We are looking for supergravity solutions dual to BPS states constructed from linear combinations of the operators

$$\mathcal{O}(q, r) = \mathrm{Tr}\big(Z_1^q\big)\mathrm{Tr}\big(Z_2^q\big)\mathrm{Tr}\big(Z_3^r\big) + \cdots \tag{3.1}$$

for constant q, where the meaning of the dots has been explained in the previous two sections. The geometries will thus be invariant under $SU(2)_L \times SO(4)_{KK}$ as defined in the previous section and invariant but charged under the remaining $U(1)_R$. The extra non-compact time-like symmetry ($\mathbb{R}_{BPS}$ of the previous section) is associated to invariance under the transformations generated by D' in the gauge theory and will emerge naturally in our construction[2].

The most generic Ansatz consistent with these symmetries is given by

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \rho_1^2\big[(\sigma^{\hat{1}})^2 + (\sigma^{\hat{2}})^2\big] + \rho_3^2(\sigma^{\hat{3}} - A_\mu \mathrm{d}x^\mu)^2 + \tilde{\rho}^2 \mathrm{d}\tilde{\Omega}_3^2 \,. \tag{3.2}$$

where ρ_1, ρ_3, $\tilde{\rho}$, A_μ and $g_{\mu\nu}$ are functions of the four coordinates x^μ. The space is a fibration of a squashed 3-sphere (on which the $SU(2)$ left-invariant 1-forms $\sigma^{\hat{a}}$ are defined) and a round 3-sphere (on which the $SU(2)$ left-invariant 1-forms $\sigma^{\tilde{a}}$ are defined) over a four dimensional manifold.
The left invariant 1-forms are given by

$$\begin{array}{ll} \sigma^{\hat{1}} = -\frac{1}{2}(\cos\hat{\psi}\, d\hat{\theta} + \sin\hat{\psi}\, \sin\hat{\theta}\, d\hat{\phi}) & \sigma^{\tilde{1}} = -\frac{1}{2}(\cos\tilde{\psi}\, d\tilde{\theta} + \sin\tilde{\psi}\, \sin\tilde{\theta}\, d\tilde{\phi}) \\ \sigma^{\hat{2}} = -\frac{1}{2}(-\sin\hat{\psi}\, d\hat{\theta} + \cos\hat{\psi}\, \sin\hat{\theta}\, d\hat{\phi}) & \sigma^{\tilde{2}} = -\frac{1}{2}(-\sin\tilde{\psi}\, d\tilde{\theta} + \cos\tilde{\psi}\, \sin\tilde{\theta}\, d\tilde{\phi}) \\ \sigma^{\hat{3}} = -\frac{1}{2}(d\hat{\psi} + \cos\hat{\theta}\, d\hat{\phi}) & \sigma^{\tilde{3}} = -\frac{1}{2}(d\tilde{\psi} + \cos\tilde{\theta}\, d\tilde{\phi}) \end{array} \tag{3.3}$$

and satisfy the relations

$$\begin{aligned} \mathrm{d}\sigma^{\hat{i}} &= \epsilon_{\hat{i}\hat{j}\hat{k}}\sigma^{\hat{j}} \wedge \sigma^{\hat{k}} \\ \mathrm{d}\sigma^{\tilde{i}} &= \epsilon_{\tilde{i}\tilde{j}\tilde{k}}\sigma^{\tilde{j}} \wedge \sigma^{\tilde{k}} \,. \end{aligned} \tag{3.4}$$

With this normalization the metric on the unit radius round three sphere is given by

$$d\Omega_3^2 = (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 \,, \tag{3.5}$$

[2]See appendix B for details.

with σ^a being either $\sigma^{\hat{a}}$ or $\sigma^{\tilde{a}}$.
We choose our "d-bein" to be

$$e^m = \varepsilon^m{}_\mu \mathrm{d}x^\mu \tag{3.6}$$

$$e^{\hat{a}} = \begin{cases} \rho_1 \sigma^{\hat{a}} & a = 1,2 \\ \rho_3(\sigma^{\hat{3}} - A_\mu \mathrm{d}x^\mu) & a = 3 \end{cases} \tag{3.7}$$

$$e^{\tilde{a}} = \tilde{\rho}\sigma^{\tilde{a}} \tag{3.8}$$

Since we are looking for the geometric dual to operators which involve only scalar fields in the gauge theory, the only possible non-zero Ramond-Ramond field strength is the five form $F_{(5)}$ and the dilaton is assumed to be constant. The most generic Ansatz for the five form which is invariant under the given symmetries is:

$$\begin{aligned} F_{(5)} = 2\left(\tilde{G}_{mn} e^m \wedge e^n + \tilde{V}_m e^m \wedge e^{\hat{3}} + \tilde{g} e^{\hat{1}} \wedge e^{\hat{2}}\right) \wedge \tilde{\rho}^3 \mathrm{d}\tilde{\Omega}_3 + \\ 2\left(-G_{pq} e^p \wedge e^q \wedge e^{\hat{1}} \wedge e^{\hat{2}} \wedge e^{\hat{3}} + \star_4 \tilde{V} \wedge e^{\hat{1}} \wedge e^{\hat{2}} - \star_4 \tilde{g} \wedge e^{\hat{3}}\right), \end{aligned} \tag{3.9}$$

where

$$G_{mn} = \frac{1}{2}\epsilon_{mnpq}\tilde{G}^{mn} \tag{3.10}$$

$$\star_4 \tilde{V} = \frac{1}{3!}\epsilon_{mnpq}\tilde{V}^m e^n \wedge e^p \wedge e^q \tag{3.11}$$

$$\star_4 \tilde{g} = \tilde{g} e^0 \wedge e^1 \wedge e^2 \wedge e^3 \tag{3.12}$$

The Bianchi identity $\mathrm{d}F_{(5)} = 0$ implies:

$$\mathrm{d}\big(\tilde{G}\tilde{\rho}^3 - \tilde{V} \wedge A\rho_3\tilde{\rho}^3\big) = 0 \tag{3.13}$$

$$\tilde{V} = \frac{1}{2}\frac{1}{\rho_3\tilde{\rho}^3}\mathrm{d}(\tilde{g}\rho_1^2\tilde{\rho}^3) \tag{3.14}$$

$$\mathrm{d}\big(G\rho_1^2\rho_3\big) = 0 \tag{3.15}$$

$$\mathrm{d}\big(G\rho_1^2\rho_3 \wedge A + \star_4\tilde{V}\big) - 2 \star_4 \tilde{g} = 0\,. \tag{3.16}$$

Since we are looking for the dual of BPS states, the background should preserve a fraction of the supersymmetry and so there should exist a supersymmetry parameter ψ such that the gravitino variation vanishes:

$$\delta\chi_M = \nabla_M\psi + \frac{\mathrm{i}}{480}F_{M_1M_2M_3M_4M_5}\Gamma^{M_1M_2M_3M_4M_5}\Gamma_M\psi = 0\,. \tag{3.17}$$

The Bianchi identity and the existence of the spinor ψ are sufficient for our supergravity background to satisfy the full equations of motion of type IIB Supergravity.
The existence of the spinor ψ is also sufficient to express the complete solution in the following form:

$$\begin{aligned} \mathrm{d}s^2 = -h^{-2}(\mathrm{d}t + V_i\mathrm{d}x^i)^2 + h^2\frac{\rho_1^2}{\rho_3^2}(T^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + \mathrm{d}y^2) + \tilde{\rho}^2\mathrm{d}\tilde{\Omega}_3^2 + \\ + \rho_1^2\big(\hat{\sigma}_1^2 + \hat{\sigma}_2^2\big) + \rho_3^2(\hat{\sigma}_3 - A_t\mathrm{d}t - A_i\mathrm{d}x^i)^2 \end{aligned} \tag{3.18}$$

where the coordinate y is the product of two radii

$$y = \rho_1 \tilde{\rho} > 0\,, \tag{3.19}$$

and the function h is given by

$$h^{-2} = \tilde{\rho}^2 + \rho_3^2 (1 + A_t)^2\,. \tag{3.20}$$

The vector ∂_t is the Killing vector which generates the extra non-compact timelike $U(1)$ and thus all the entries of the metric depend only on (x^1, x^2, y), where y is constrained to be positive. They can be expressed in terms of four independent functions:

$$m, n, p, T$$

as follows:

$$\begin{array}{lll} \rho_1^4 = \frac{mp+n^2}{m} y^4 & \rho_3^4 = \frac{p^2}{m(mp+n^2)} & \tilde{\rho}^4 = \frac{m}{mp+n^2} \\ h^4 = \frac{mp^2}{mp+n^2} & A_t = \frac{n-p}{p} & A_i = A_t V_i - \frac{1}{2}\epsilon_{ij}\partial_j \ln T \end{array} \tag{3.21}$$

and

$$\mathrm{d}V = -y \star_3 [\mathrm{d}n + (nD + 2ym(n-p) + 2n/y)\mathrm{d}y] \tag{3.22}$$

$$\partial_y \ln T = D \tag{3.23}$$

$$D \equiv 2y(m + n - 1/y^2)\,, \tag{3.24}$$

where $\star_3$ indicates the Hodge dual in the three dimensional diagonal metric

$$\mathrm{d}s_3^2 = T^2 \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j + \mathrm{d}y^2\,. \tag{3.25}$$

The various four-dimensional forms from which the 5-form field strength is constructed are

$$\tilde{g} = \frac{1}{4\tilde{\rho}} \left[1 - \frac{\rho_3^2}{\rho_1^2}(1 + A_t)\right] \tag{3.26}$$

$$\tilde{V} = \frac{1}{2} \frac{1}{\rho_3 \tilde{\rho}^3} \mathrm{d}(\tilde{g} \rho_1^2 \tilde{\rho}^3) \tag{3.27}$$

$$G\rho_1^2 \rho_3 = \mathrm{d}B_t \wedge (\mathrm{d}t + V_i \mathrm{d}x^i) + B_t \mathrm{d}V + \mathrm{d}\hat{B} \tag{3.28}$$

$$\tilde{G}\tilde{\rho}^3 = \frac{1}{2} g \rho_1^2 \tilde{\rho}^3 \mathrm{d}A + \mathrm{d}\tilde{B}_t \wedge (\mathrm{d}t + V_i \mathrm{d}x^i) + \tilde{B}_t \mathrm{d}V + \mathrm{d}\hat{\tilde{B}}\,, \tag{3.29}$$

where

$$\begin{aligned} \tilde{B}_t &= -\frac{1}{16} y^2 \frac{n - 1/y^2}{p} \\ \mathrm{d}\hat{\tilde{B}} &= -\frac{1}{16} y^3 \star_3 [\mathrm{d}m + 2mD\,\mathrm{d}y] \\ B_t &= -\frac{1}{16} y^2 \frac{n}{m} \\ \mathrm{d}\hat{B} &= \frac{1}{16} y^3 \star_3 [\mathrm{d}p + 4yn(p-n)\mathrm{d}y]\,. \end{aligned} \tag{3.30}$$

Differential equations

The Bianchi identities and the integrability condition for the equation (3.22) give

$$\begin{cases} \mathrm{d}\mathrm{d}V = 0 \\ \mathrm{d}\mathrm{d}\hat{\tilde{B}} = 0 \\ \mathrm{d}\mathrm{d}\hat{B} = 0 \end{cases} . \tag{3.31}$$

These three conditions together with (3.24) give a system of nonlinear coupled elliptic differential equations

$$\begin{aligned} & y^3(\partial_1^2+\partial_2^2)n + \partial_y\left(y^3T^2\partial_y n\right) + y^2\partial_y\left[T^2\left(yDn+2y^2m(n-p)\right)\right] + 4y^2DT^2n = 0 \\ & y^3(\partial_1^2+\partial_2^2)m + \partial_y\left(y^3T^2\partial_y m\right) + \partial_y\left(y^3T^2 2mD\right) = 0 \\ & y^3(\partial_1^2+\partial_2^2)p + \partial_y\left(y^3T^2\partial_y p\right) + \partial_y\left[y^3T^2 4ny(n-p)\right] = 0 \\ & \partial_y \ln T = D\,. \end{aligned} \tag{3.32}$$

A solution to these equations is determined by a set of boundary conditions at infinity (large values of y, x^i) and on the plane $y = 0$; they should be chosen in such a way as to give a non-singular geometry asymptotic to $AdS_5 \times S^5$. Due to the non-linearity of the equations the relationship between boundary conditions and non-singular solutions with $AdS_5 \times S^5$ asymptotics is difficult to control. This set of boundary conditions may be regarded as a parametrisation of the space of solutions to our problem.

The LLM limit

The LLM solutions are clearly a subset of ours. They are specified by the additional constraints,

$$n = p = \frac{1}{y^2} - m = \frac{1/2 - z}{y^2} \quad T = 1\,. \tag{3.33}$$

In this case we have

$$D = 0 \quad \rho_1 = \rho_3 = \rho \quad A_t = 0 \quad T = 1 \tag{3.34}$$

and the three second order equations collapse to one single linear equation

$$y^3(\partial_1^2+\partial_2^2)n + \partial_y\left(y^3T^2\partial_y n\right)\,. \tag{3.35}$$

As this equation is linear it has been possible to completely identify the boundary conditions at $y = 0$ and at infinity that give rise to regular asymptotically $AdS_5 \times S^5$ geometries [1, 29]. This set of boundary conditions can be directly identified with the classical phase space of the dual states in the free fermion picture.

4. Asymptotics and charges

In this section we discuss asymptotic solutions to the differential equations of the previous section wich give $AdS_5 \times S^5$ asymptotics [3]. We solve the equations to third order in an expansion for large values of y, x^1, x^2.

[3] A study of more general boundary conditions at $y = 0$ will be presented in [30].

We can identify the boundary conditions at infinity by comparing the leading order of this expansion to the same order of LLM, requiring in particular $AdS_5 \times S^5$ asymptotics. The first corrections to the $AdS_5 \times S^5$ geometry capture the global $U(1)$ charges under the gauge fields arising in the Kaluza Klein reduction of IIB supergravity over S^5. We will show that the solutions support non-vanishing fluxes for the the KK gauge fields associated to two of the three Cartan generators of the $SO(6)$ isometry of S^5. In the dual gauge theory picture these generators map to the R-symmetry generators $L_{5,6}$ and $L_{1,2} + L_{3,4}$.

It is not hard to see that the following expressions for our functions

$$\begin{aligned} m &\sim \frac{1}{y^2} - \frac{p_1}{R^4} \\ n &\sim \frac{p_1}{R^4} \\ p &\sim \frac{p_1}{R^4} \\ T &\sim 1 \end{aligned} \tag{4.1}$$

satisfy the equations at leading order for large R, with (R, θ, ϕ) polar coordinates in the (x_1, x_2, y) space and p_1 is a constant parameter. We have also, to the same order,

$$V_\phi \sim \frac{p_1 \cos^2\theta}{R^2} \qquad V_r \sim O\left(\frac{1}{R^4}\right), \tag{4.2}$$

with $r^2 = x_1^2 + x_2^2$, $r = R\cos\theta$ and $y = R\sin\theta$.
Defining

$$\begin{aligned} \tilde{R} &= R/\sqrt{p_1} \\ \tilde{\phi} &= \phi - t \end{aligned} \tag{4.3}$$

we get

$$\mathrm{d}s^2 = \sqrt{p}_1 \left(-\tilde{R}^2 \mathrm{d}t^2 + \frac{\mathrm{d}\tilde{R}^2}{\tilde{R}^2} + \tilde{R}^2 \mathrm{d}\tilde{\Omega}_3^2 + \mathrm{d}\theta^2 + \cos^2\theta \mathrm{d}\tilde{\phi}^2 + \sin^2\theta \mathrm{d}\hat{\Omega}_3^2 \right) \tag{4.4}$$

which is $AdS_5 \times S^5$ in Poincare coordinates. The parameter p_1 and the radius L of AdS_5 are related by

$$L^2 = \sqrt{p_1}\,. \tag{4.5}$$

We recall here the expression for the left-invariant one forms

$$\begin{aligned} \sigma^{\hat{1}} &= -\tfrac{1}{2}(\cos\hat{\psi}\, d\hat{\theta} + \sin\hat{\psi}\, \sin\hat{\theta}\, d\hat{\phi}) \\ \sigma^{\hat{2}} &= -\tfrac{1}{2}(-\sin\hat{\psi}\, d\hat{\theta} + \cos\hat{\psi}\, \sin\hat{\theta}\, d\hat{\phi}) \\ \sigma^{\hat{3}} &= -\tfrac{1}{2}(d\hat{\psi} + \cos\hat{\theta}\, d\hat{\phi})\,. \end{aligned} \tag{4.6}$$

The metric on the unit radius round three sphere $\mathrm{d}\hat{\Omega}_3$ is

$$\mathrm{d}\hat{\Omega}_3^2 = \left(\sigma^{\hat{1}}\right)^2 + \left(\sigma^{\hat{2}}\right)^2 + \left(\sigma^{\hat{3}}\right)^2 = \frac{1}{4}\left(\mathrm{d}\hat{\theta}^2 + \mathrm{d}\hat{\phi}^2 + \mathrm{d}\hat{\psi}^2 + 2\cos\hat{\theta}\mathrm{d}\hat{\psi}\mathrm{d}\hat{\phi}\right). \tag{4.7}$$

We can transform it into the more conventional form

$$\mathrm{d}\hat{\Omega}_3^2 = \mathrm{d}\omega^2 + \cos^2\omega\,\mathrm{d}\phi_1^2 + \sin^2\omega\,\mathrm{d}\phi_2^2 \tag{4.8}$$

where

$$\omega = \frac{\hat{\theta}}{2} \qquad \phi_1 = \frac{\hat{\psi}+\hat{\phi}}{2} \qquad \phi_2 = \frac{\hat{\psi}-\hat{\phi}}{2}\,. \tag{4.9}$$

We will now consider the next two orders in the asymptotic expansion of our functions and solve the differential equations. For the sake of simplicity we will assume that ∂_ϕ is also a Killing vector of our solutions. Despite this simplifying assumption, in general the solutions will still be charged under the corresponding KK gauge field. From the geometric point of view this means that the solutions are generically stationary. On the gauge theory side, this choice corresponds to looking for duals of linear combinations of states which have all the same L_{56} charge[4] and are thus constructed from linear combinations of $\mathcal{O}(q,r)$ at fixed q and r.

We thus assume the following expansion of our functions:

$$\begin{aligned} m &\sim \frac{1}{y^2} - \frac{p_1}{R^4} + \frac{m_2(\theta)}{R^6} + \frac{m_3(\theta)}{R^8} \\ n &\sim \frac{p_1}{R^4} + \frac{n_2(\theta)}{R^6} + \frac{n_3(\theta)}{R^8} \\ p &\sim \frac{p_1}{R^4} + \frac{p_2(\theta)}{R^6} + \frac{p_3(\theta)}{R^8} \\ T &\sim 1 + \frac{t_1(\theta)}{R^2} + \frac{t_2(\theta)}{R^4}\,. \end{aligned} \tag{4.10}$$

Recalling that $D = 2y(m+n-1/y^2)$, the equation

$$\partial_y \ln T = D \tag{4.11}$$

implies that

$$t_1(\theta) = 0\,. \tag{4.12}$$

Moreover we note that

$$V_\phi \sim \frac{p_1\cos^2\theta}{R^2} + \frac{V_2(\theta)}{R^4} \qquad V_r = 0\,. \tag{4.13}$$

With a suitable coordinate transformation

$$\begin{cases} R = \sqrt{p_1}\tilde{R} + \frac{h_1(\tilde{\theta})}{\tilde{R}} \\ \theta = \tilde{\theta} + \frac{g_1(\tilde{\theta})}{\tilde{R}^2} \\ \phi = \tilde{\phi} + t \end{cases} \tag{4.14}$$

[4]The analog of this choice in the LLM picture would be to consider solutions seeded by rotationally symmetric configurations of bubbles on the $y = 0$ plane.

it should be possible to bring the metric to the following form:

$$\begin{aligned}
\mathrm{d}s^2 = \Omega(\tilde{R},\tilde{\theta})L^2\left[-(1+\tilde{R}^2-\frac{\tilde{R}_0^2}{\tilde{R}^2})\mathrm{d}t^2+\frac{\mathrm{d}\tilde{R}^2}{\tilde{R}^2}(1-\frac{1}{\tilde{R}^2})+\tilde{R}^2\mathrm{d}\tilde{\Omega}_3^2\right]+ \\
+L^2\Big[g_{\tilde{\theta}\tilde{\theta}}\mathrm{d}\tilde{\theta}^2+g_{\tilde{\phi}\tilde{\phi}}\cos^2\tilde{\theta}\left(\mathrm{d}\tilde{\phi}+\frac{J}{\tilde{R}^2}\mathrm{d}t\right)^2+ \\
+g_{\omega\omega}\sin^2\tilde{\theta}\mathrm{d}\omega^2+g_{\phi\phi}\sin^2\tilde{\theta}\Big(\cos^2\omega(\mathrm{d}\phi_1-\frac{Q}{\tilde{R}^2}\mathrm{d}t)^2+\sin^2\omega(\mathrm{d}\phi_2-\frac{Q}{\tilde{R}^2}\mathrm{d}t)^2\Big)\Big] \quad (4.15)
\end{aligned}$$

up to subleading corrections. To the leading order the metric components $g_{\tilde{\theta}\tilde{\theta}} = g_{\tilde{\phi}\tilde{\phi}} = g_{\omega\omega} = g_{\phi\phi} = 1$ and reproduce S^5. The constants J and Q are proportional to the total flux of the $U(1)$ gauge fields arising from the KK reduction of the supergravity over S^5. In particular Q is the total charge of the solutions under both the gauge field associated with coordinate transformations generated by $\lambda(\xi)\partial_{\phi_1}$ and $\mu(\xi)\partial_{\phi_2}$ (being ξ coordinates in the AdS_5 factor); these are dual respectively to the $J_1 = L_{1,2}$ and $J_2 = L_{3,4}$ R-symmetry generators. For this reason the expected BPS relation is

$$M = \frac{\pi L^2}{4G_5}(|J|+2|Q|)\,. \quad (4.16)$$

The conformal factor $\Omega(\tilde{R},\tilde{\theta})$ satisfies $\Omega(\tilde{R}=\infty,\tilde{\theta}) = 1$ and contains terms up to order $\tilde{R}^{-4}$. The mass of the excitations over the AdS_5 vacuum is given by

$$M = \frac{3\pi L^2}{8G_5}\tilde{R}_0 \quad (4.17)$$

where G_5 is the five-dimensional Newton constant[5]. We recall now the expression for the metric:

$$\begin{aligned}
\mathrm{d}s^2 = -h^{-2}(\mathrm{d}t^2+V_\phi\mathrm{d}\phi)^2+h^2\frac{\rho_1^2}{\rho_3^2}(T^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j+\mathrm{d}y^2)+ \\
+\tilde{\rho}^2\mathrm{d}\Omega_3^2+\rho_1^2\left[\left(\sigma^{\hat{1}}\right)^2+\left(\sigma^{\hat{2}}\right)^2\right]+\rho_3^2(\sigma^{\hat{3}}-A_t\mathrm{d}t-A_\phi\mathrm{d}\phi)^2 = \\
=g_{tt}\mathrm{d}t^2+g_{RR}\mathrm{d}R^2+\tilde{\rho}^2\mathrm{d}\tilde{\Omega}_3^2+2g_{\theta R}\mathrm{d}\theta\mathrm{d}R+ \\
+g_{t\tilde{\phi}}\mathrm{d}t\mathrm{d}\tilde{\phi}+g_{t\hat{3}}\mathrm{d}t\sigma^{\hat{3}}+ \\
+g_{\theta\theta}\mathrm{d}\theta^2+g_{\tilde{\phi}\tilde{\phi}}\mathrm{d}\tilde{\phi}^2+g_{\tilde{\phi}\hat{3}}\mathrm{d}\tilde{\phi}\,\sigma^{\hat{3}}+\rho_1^2\left[\left(\sigma^{\hat{1}}\right)^2+\left(\sigma^{\hat{2}}\right)^2\right]+\rho_3^2\left(\sigma^{\hat{3}}\right)^2 \quad (4.18)
\end{aligned}$$

[5]This approach follows the one in [1]. A more precise and detailed approach can be taken following e.g. the work in [40]

with

$$
\begin{aligned}
g_{tt} &= -h^{-2}(1+V_\phi)^2 + h^2\frac{\rho_1^2}{\rho_3^2}r^2T^2 + \rho_3^2(A_\phi+A_t)^2\\
g_{RR} &= h^2\frac{\rho_1^2}{\rho_3^2}(\sin^2\theta + T^2\cos^2\theta)\\
g_{\theta R} &= h^2\frac{\rho_1^2}{\rho_3^2}R\sin\theta\cos\theta(1-T^2)\\
g_{t\tilde{\phi}} &= -h^{-2}(1+V_\phi)V_\phi + h^2\frac{\rho_1^2}{\rho_3^2}r^2T^2 + \rho_3^2(A_t+A_\phi)A_\phi\\
g_{t\hat{3}} &= -\rho_3^2(A_t+A_\phi)\\
g_{\theta\theta} &= h^2\frac{\rho_1^2}{\rho_3^2}R^2(\cos^2\theta + T^2\sin^2\theta)\\
g_{\tilde{\phi}\tilde{\phi}} &= -h^{-2}V_\phi^2 + h^2\frac{\rho_1^2}{\rho_3^2}r^2T^2 + \rho_3^2A_\phi^2\\
g_{\tilde{\phi}\hat{3}} &= \rho_3^2A_\phi
\end{aligned}
\tag{4.19}
$$

We can now derive the Q charge of our solutions. Using the definition (4.6) and the coordinate transformation (4.9) we get

$$
Q = -\frac{g_{t\hat{3}}}{g_{\hat{3}\hat{3}}}\tilde{R}^2 = (A_t + A_\phi)\,. \tag{4.20}
$$

We note that $A_t = (n-p)/p = O(1/R^2)$ and $A_\phi = A_tV_\phi + \frac{1}{2}r\partial_r \ln T = O(1/R^4)$ and thus the leading behaviour of the r.h.s. is determined by A_t and we have

$$
Q = \frac{n_2(\theta) - p_2(\theta)}{p_1^2}\,. \tag{4.21}
$$

Using these relations we can solve the equations (3.32) up to second order in $\frac{1}{R^2}$ and demanding that the solutions are regular we find

$$
\begin{cases}
p_2(\theta) = d(3\cos^2\theta - 1)\\
n_2(\theta) = p_2(\theta) + p_1^2Q\\
m_2(\theta) = -p_2(\theta) - 2p_1^2Q\\
V_2(\theta) = \dfrac{1}{2}\cos^2\theta\big[(Qp_1^2 - d + 3d\cos(2\theta)\big]
\end{cases}
\tag{4.22}
$$

where d is a generic real integration constant. The J charge is given by

$$
J = \frac{g_{t\tilde{\phi}}}{g_{\tilde{\phi}\tilde{\phi}}}\,\tilde{R}^2 = \frac{d}{p_1^2} - 1 - Q. \tag{4.23}
$$

The conserved charges Q and J can be also obtained by evaluating Komar integrals associated with the Killing vectors $\hat{\Sigma}_3$ (the dual vector field to $\hat{\sigma}_3$) and $\frac{\partial}{\partial\phi}$ respectively.

We will now solve the equations to the next order and find the transformation (4.14) that brings the metric to the form (4.15) enabling us to check that the BPS mass formula

$$M = \frac{\pi L^2}{4G_5}(|J| + 2|Q|) \tag{4.24}$$

is satisfied.

We have

$$g_{\tilde{\theta}\tilde{R}} = [h_1'(\tilde{\theta}) - 2\sqrt{p_1}g_1(\tilde{\theta})]\frac{1}{\tilde{R}^3} \tag{4.25}$$

which fixes

$$g_1(\tilde{\theta}) = \frac{h_1'(\tilde{\theta})}{2\sqrt{p_1}}\,. \tag{4.26}$$

We are not really interested in the conformal factor $\Omega(\tilde{R}, \tilde{\theta})$ and we thus proceed to the calculation of the ratio

$$\frac{g_{\tilde{R}\tilde{R}}}{\tilde{\rho}^2} = \frac{1}{\tilde{R}^4} + \frac{d(3\cos^2\tilde{\theta} - 1) - 6p_1^{3/2}h_1(\tilde{\theta})}{p_1^2} \tag{4.27}$$

which should satisfy the equation

$$\frac{g_{\tilde{R}\tilde{R}}}{\tilde{\rho}^2} = \frac{1}{\tilde{R}^4} - \frac{1}{\tilde{R}^6}\,. \tag{4.28}$$

This requirement gives immediately,

$$h_1(\tilde{\theta}) = \frac{p_1^2 + d(3\cos^2\tilde{\theta} - 1)}{6p_1^{3/2}}\,. \tag{4.29}$$

Using this relation we obtain

$$\frac{g_{tt}}{\tilde{\rho}^2} = -1 - \frac{1}{\tilde{R}^2} + \frac{2}{3}\left(\frac{d}{p_1^2} - 1 - 3Q\right)\frac{1}{\tilde{R}^4} \tag{4.30}$$

which gives

$$\tilde{R}_0 = \frac{2}{3}(J - 2Q) \tag{4.31}$$

and thus

$$M = \frac{3\pi L^2}{8G_5}\tilde{R}_0 = \frac{\pi L^2}{4G_5}(J - 2Q)\,. \tag{4.32}$$

This should be compared to

$$M = \frac{\pi L^2}{4G_5}(|J| + 2|Q|)\,, \tag{4.33}$$

which apparently requires that $J > 0$ and $Q < 0$. Up to now, J and Q have appeared in the solution to the differential equations as constants of integration. As such, they can take any real value. Constraints on their possible values should come from a global analysis of

the solutions [6]. Indeed given the leading behaviour at large R, these subleading corrections should be completely determined by the boundary conditions at $y = 0$. Unfortunately we are not able to express these charges in terms of the data at $y = 0$ plane which could have allowed us to establish the above bounds on J and Q. As a matter of comparison, in the LLM construction only the J charge is present and its value is determined by a set of integrals performed on the $y = 0$ plane. In that case, the bound $J > 0$ is trivially imposed by the specific type of boundary conditions at $y = 0$.

5. Conclusions and perspectives

In this paper we have extended to the 1/8 BPS case the construction of [1]. Due to the reduced amount of symmetry of our background the expressions we find turn out to be rather more complex; in particular the differential equations which determine the background are highly non linear. We performed an asymptotic analysis for large values of R and were able to show that solutions with the desired asymptotics and regularity exist in this limit. Of course, a satisfactory understanding of the boundary conditions at $y = 0$ which lead to non-singular solutions is necessary in order to connect the geometry of the supergravity solutions to the phase space of the quantum mechanical system arising from the dual gauge theory on $\mathbb{R} \times S^3$. In particular it would be very interesting to understand the relationship between our construction and the work of [12, 13, 15, 14]. Once the space of solutions is understood from the supergravity point of view one could proceed to its quantisation by a procedure like that presented in [31, 32].

Our solutions have a non empty intersection with the solutions described in [11, 10] and in [9]. It would be interesting to find the exact dictionary between different descriptions of the same solutions in order to better clarify the role of the boundary conditions at $y = 0$ and to try to recast the differential equations in a more tractable fashion. 1/4 BPS solutions can be obtained from the general setting that we have presented by imposing some additional constraints on the four scalar functions [30]. Some of the so-called superstar geometries in [33] are also contained in our description. These solutions are known to have singularities and it is possible to identify the boundary conditions at $y = 0$ that are responsible for them. With a more detailed understanding of boundary conditions which give rise to non-singular solutions, and their relation to the CFT, one may better understand the resolution of the singularities in a manner similar to that of [34, 29, 35]. Finally different types of boundary conditions at large R can be studied. Indeed one can find solutions with asymptotics of the form $AdS_5 \times Y^{p,q}$: such geometries correspond to 1/2 BPS operators in the $\mathcal{N} = 1$ superconformal quiver gauge theories [36].

[6] As in the LLM case, the sign of J is correlated with the relative chirality of the Killing spinor with respect to the two $SO(4)$'s. From the gauge theory side, as follows from the discussion at the end of section 2, the sign of Q is correlated with the $U(1)_R$ charge of the Killing spinor. As it emerges from the detailed analysis of appendix B, this charge is captured by the eigenvalue s with respect to a Pauli matrix $\sigma_{\hat{3}}$. In our analysis we have set for definiteness $s = +1$. Had we chosen $s = -1$, Q would have been positive.

JHEP05(2007)030

Acknowledgments

This work was supported in part by the European Commission under the RTN contract MRTN-CT-2004-503369. M.O'L. would like to thank the ICTP for hospitality during the course of this work. The work of G.M. is supported in part by the EC under the contract MRTN-CT-2004-005104.

A. Conventions

We set up our conventions for the wedge product of 1-forms

$$\alpha_1 \wedge \cdots \wedge \alpha_n = \frac{1}{n!} \sum_i \sigma(i) \alpha_{i(1)} \otimes \cdots \otimes \alpha_{i(n)} \tag{A.1}$$

where the sum is over the $n!$ permutations i and $\sigma(i)$ is the parity of the permutation.

An n-form α in a d dimensional space ($\alpha \in \Lambda_n$) is given by

$$\alpha = \bar{\alpha}_{\mu_1 \cdots \mu_n} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} = \frac{1}{n!} \alpha_{\mu_1 \cdots \mu_n} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \tag{A.2}$$

with $\alpha_{\mu_1 \cdots \mu_n}$ the complete antisymmetrization of $\bar{\alpha}_{\mu_1 \cdots \mu_n}$.

When a metric is present we can introduce the Hodge dual

$$\star : \Lambda_n \to \Lambda_{d-n} \tag{A.3}$$

Given a d-bein of the metric $\{e^m\}_{m=1,\cdots d}$,

$$\star e^{m_1} \wedge \cdots \wedge e^{m_n} = \frac{1}{(d-n)!} \epsilon^{m_1, \ldots m_n, m_{n+1}, \cdots, m_d} e_{m_{n+1}} \wedge \cdots \wedge e_{m_d} \tag{A.4}$$

where indices are lowered with the tangent space metric. From this definition it follows that

$$\begin{aligned} \star\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} &= \star g^{\mu_1 \mu_1'} \cdots g^{\mu_n \mu_n'} e_{m_1 \mu_1'} \cdots e_{m_n \mu_n'} e^{m_1} \wedge \cdots \wedge e^{m_n} = \\ &= g^{\mu_1 \mu_1'} \cdots g^{\mu_n \mu_n'} e_{m_1 \mu_1'} \cdots e_{m_n \mu_n'} \frac{1}{(d-n)!} \epsilon^{m_1, \cdots m_n, m_{n+1}, \cdots, m_d} e_{m_{n+1}} \wedge \cdots \wedge e_{m_d} = \\ &= \frac{1}{(d-n)!} \sqrt{g} g^{\mu_1 \mu_1'} \cdots g^{\mu_n \mu_n'} \epsilon_{\mu_1', \cdots \mu_n', \mu_{n+1}', \cdots, \mu_d'} \mathrm{d}x^{\mu_{n+1}'} \wedge \cdots \wedge \mathrm{d}x^{\mu_d'} . \end{aligned} \tag{A.5}$$

The exterior derivative of a 1-form is defined by

$$\beta = \mathrm{d}\alpha = \partial_\mu \alpha_\nu \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2} (\partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{A.6}$$

or in terms of components $\beta_{\mu\nu} = \partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu$. The generalization to any n-form is given by

$$\begin{aligned} \beta = \mathrm{d}\alpha &= \frac{1}{n!} \partial_\mu \alpha_{\nu_1 \cdots \nu_n} \mathrm{d}x^\mu \wedge \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_n} = \\ &= \frac{1}{(n+1)!} \partial_{[\mu} \alpha_{\nu_1 \cdots \nu_n]} \mathrm{d}x^\mu \wedge \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_n} = \frac{1}{(n+1)!} \beta_{\nu_1 \cdots \nu_n} \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_n} \end{aligned} \tag{A.7}$$

where now $\beta_{\mu\nu_1\cdots\nu_n} = \partial_{[\mu}\alpha_{\nu_1\cdots\nu_n]}$ and square brackets indicate antisymmetrization without normalization.

The torsionless spin connection 1-form is defined by the structure equation:

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0\,. \tag{A.8}$$

Requiring metricity of the connection

$$\omega_{ab} = -\omega_{ba} \tag{A.9}$$

allows us to explicitly express ω_{ab} in terms of the d-bein (E_a are the inverse d-bein vector fields, defined by $e^a \cdot E_b = \delta^a{}_a$),

$$\begin{aligned}
\omega_{ab} = -\mathrm{d}e_a \cdot E_b + \mathrm{d}e_b \cdot E_a + \frac{1}{2}\left(e^c \cdot [E_a, E_b]\right) e_c = \\
= \Big[-\frac{1}{2}\left(\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}\right) E^\nu{}_b + \frac{1}{2}\left(\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}\right) E^\nu{}_a + \\
+ \frac{1}{2} e^c{}_\rho \left(E^\nu{}_a \partial_\nu E^\rho{}_b - E^\nu{}_b \partial_\nu E^\rho{}_a \right) e_{c\mu} \Big] \mathrm{d}x^\mu = \\
= \Big[-\frac{1}{2}\left(\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}\right) E^\nu{}_b + \frac{1}{2}\left(\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}\right) E^\nu{}_a + \\
- \frac{1}{2} E^\nu{}_a \left(\partial_\nu e^c{}_\rho - \partial_\rho e^c{}_\nu \right) E^\rho{}_b e_{c\mu} \Big] \mathrm{d}x^\mu = \\
= -\mathrm{d}e_a \cdot E_b + \mathrm{d}e_b \cdot E_a - \left(E_a \cdot \mathrm{d}e^c \cdot E_b\right) e_c
\end{aligned} \tag{A.10}$$

where in going from the second to the third line we have used

$$0 = \partial_\mu \eta_{ab} = \partial_\mu \left(e_{a\nu} E^\nu{}_b\right) = \left(\partial_\mu e_{a\nu}\right) E^\nu{}_b + e_{a\nu}\left(\partial_\mu E^\nu{}_b\right)\,. \tag{A.11}$$

This is an explicit realization of the identity

$$V \cdot \mathrm{d}\alpha \cdot W = \frac{1}{2}\mathrm{d}(\alpha \cdot W) \cdot V - \frac{1}{2}\mathrm{d}(\alpha \cdot V) \cdot W - \frac{1}{2}\alpha \cdot [V, W]\,, \tag{A.12}$$

which holds for any one form α and any pair of vector fields V, W.

The covariant derivative of a spinor is given by

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{4}\omega_{ab\mu}\Gamma^a\Gamma^b\psi\,. \tag{A.13}$$

Group manifolds

Consider a Lie algebra of vector fields on a d-dimensional group manifold. It is a d dimensional vector space of vector fields satisfying

$$[E_a, E_b] = f_{ab}{}^c E_c\,. \tag{A.14}$$

The exterior derivative of the dual one forms is given by

$$\mathrm{d}e^c = \frac{1}{2}\alpha_{ab}{}^c e^a \wedge e^b \tag{A.15}$$

These are the Maurer Cartan 1-forms. Indeed, we have

$$E_a \cdot \mathrm{d}e^c \cdot E_b = \frac{1}{2}\alpha_{ab}{}^c \tag{A.16}$$

and according to (A.12)

$$E_a \cdot \mathrm{d}e^c \cdot E_b = -\frac{1}{2}e^c \cdot [E_a, E_b] \tag{A.17}$$

which give

$$\alpha_{ab}{}^c = -f_{ab}{}^c \,. \tag{A.18}$$

The Lie derivative of a 1-form is defined by

$$(\mathcal{L}_J\omega)\cdot K = \partial_K(\omega\cdot K) - \omega\cdot[J,K] \tag{A.19}$$

and thus

$$\mathcal{L}_{E_a}e^c = -f_{ab}{}^c e^b \tag{A.20}$$

Taking these e^a as the d-bein, the spin connection on the group manifold is given by

$$\omega_{abc} = \frac{1}{2}(-\alpha_{cba} + \alpha_{cab} + f_{abc}) = \frac{1}{2}(f_{cba} - f_{cab} + f_{abc})\,. \tag{A.21}$$

B. Reduction of the Killing spinor equations

In this appendix we present the step by step derivation of the results presented in section 3.

B.1 Metric and 5-form ansatz

The most generic Ansatz for our solutions is given by

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \rho_1^2\left[(\sigma^{\hat{1}})^2 + (\sigma^{\hat{2}})^2\right] + \rho_3^2(\sigma^{\hat{3}} - A_\mu \mathrm{d}x^\mu)^2 + \tilde{\rho}^2\mathrm{d}\tilde{\Omega}_3^2\,. \tag{B.1}$$

The space is thus made up of a fibration over a four dimensional manifold of a squashed 3-sphere (on which the $SU(2)$ left-invariant 1-forms $\sigma^{\hat{a}}$ are defined) and a round 3-sphere (on which the $SU(2)$ left-invariant 1-forms $\sigma^{\tilde{a}}$ are defined).
The left invariant 1-forms are given by

$$\begin{array}{ll}
\sigma^{\hat{1}} = -\frac{1}{2}(\cos\hat{\psi}\, d\hat{\theta} + \sin\hat{\psi}\, \sin\hat{\theta}\, d\hat{\phi}) & \sigma^{\tilde{1}} = -\frac{1}{2}(\cos\tilde{\psi}\, d\tilde{\theta} + \sin\tilde{\psi}\, \sin\tilde{\theta}\, d\tilde{\phi}) \\
\sigma^{\hat{2}} = -\frac{1}{2}(-\sin\hat{\psi}\, d\hat{\theta} + \cos\hat{\psi}\, \sin\hat{\theta}\, d\hat{\phi}) & \sigma^{\tilde{2}} = -\frac{1}{2}(-\sin\tilde{\psi}\, d\tilde{\theta} + \cos\tilde{\psi}\, \sin\tilde{\theta}\, d\tilde{\phi}) \\
\sigma^{\hat{3}} = -\frac{1}{2}(d\hat{\psi} + \cos\hat{\theta}\, d\hat{\phi}) & \sigma^{\tilde{3}} = -\frac{1}{2}(d\tilde{\psi} + \cos\tilde{\theta}\, d\tilde{\phi})
\end{array} \tag{B.2}$$

and satisfy the relations

$$\begin{aligned}
\mathrm{d}\sigma^{\hat{i}} &= \epsilon_{\hat{i}\hat{j}\hat{k}}\sigma^{\hat{j}} \wedge \sigma^{\hat{k}} \\
\mathrm{d}\sigma^{\tilde{i}} &= \epsilon_{\tilde{i}\tilde{j}\tilde{k}}\sigma^{\tilde{j}} \wedge \sigma^{\tilde{k}}\,.
\end{aligned} \tag{B.3}$$

With this normalization the metric on the unit radius round three sphere is given by

$$d\Omega_3^2 = (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 \tag{B.4}$$

with σ^a being either $\sigma^{\hat{a}}$ or $\sigma^{\tilde{a}}$.
We choose our "d-bein" to be

$$e^m = \varepsilon^m{}_\mu \mathrm{d}x^\mu \tag{B.5}$$

$$e^{\hat{a}} = \begin{cases} \rho_1 \sigma^{\hat{a}} & a = 1,2 \\ \rho_3(\sigma^{\hat{3}} - A_\mu \mathrm{d}x^\mu) & a = 3 \end{cases} \tag{B.6}$$

$$e^{\tilde{a}} = \tilde{\rho}\sigma^{\tilde{a}} \tag{B.7}$$

The only non zero Ramond-Ramond field strength is the five form $F_{(5)}$ and the dilaton is assumed to be constant. The most generic Ansatz for the five form which is invariant under the given symmetries is

$$\begin{aligned} F_{(5)} = 2\left(\tilde{G}_{mn} e^m \wedge e^n + \tilde{V}_m e^m \wedge e^{\hat{3}} + \tilde{g} e^{\hat{1}} \wedge e^{\hat{2}}\right) \wedge \tilde{\rho}^3 \mathrm{d}\tilde{\Omega}_3 + \\ 2\left(-G_{pq} e^p \wedge e^q \wedge e^{\hat{1}} \wedge e^{\hat{2}} \wedge e^{\hat{3}} + \star_4 \tilde{V} \wedge e^{\hat{1}} \wedge e^{\hat{2}} - \star_4 \tilde{g} \wedge e^{\hat{3}}\right), \end{aligned} \tag{B.8}$$

where

$$G_{mn} = \frac{1}{2}\epsilon_{mnpq}\tilde{G}^{mn} \tag{B.9}$$

$$\star_4 \tilde{V} = \frac{1}{3!}\epsilon_{mnpq}\tilde{V}^m e^n \wedge e^p \wedge e^q \tag{B.10}$$

$$\star_4 \tilde{g} = \tilde{g} e^0 \wedge e^1 \wedge e^2 \wedge e^3 . \tag{B.11}$$

The Bianchi identity $\mathrm{d}F_{(5)} = 0$ gives rise to the set of equations,

$$\mathrm{d}\big(\tilde{G}\tilde{\rho}^3 - \tilde{V} \wedge A\rho_3\tilde{\rho}^3\big) = 0 \tag{B.12}$$

$$\tilde{V} = \frac{1}{2}\frac{1}{\rho_3\tilde{\rho}^3}\mathrm{d}(\tilde{g}\rho_1^2\tilde{\rho}^3) \tag{B.13}$$

$$\mathrm{d}\big(G\rho_1^2\rho_3\big) = 0 \tag{B.14}$$

$$\mathrm{d}\big(G\rho_1^2\rho_3 \wedge A + \star_4\tilde{V}\big) - 2\star_4 \tilde{g} = 0 . \tag{B.15}$$

B.2 Spin connection and covariant derivative

The inverse d-bein is

$$E_m = \Xi^\mu{}_m \partial_\mu + A_m \Sigma^{\hat{i}}{}_{\hat{3}} \partial_{\hat{i}} \tag{B.16}$$

$$E_{\hat{a}} = \frac{1}{\rho_a}\Sigma^{\hat{i}}{}_{\hat{a}}\partial_{\hat{i}} \tag{B.17}$$

$$E_{\tilde{a}} = \frac{1}{\tilde{\rho}}\Sigma^{\tilde{i}}{}_{\tilde{a}}\partial_{\tilde{i}} , \tag{B.18}$$

where Ξ_m is the inverse vierbein of ε^m and $\Sigma_{\hat{a},\tilde{a}}$ is the inverse of $\sigma^{\hat{a},\tilde{a}}$. We will denote ten-dimensional tangent space indices by $A, B, C \ldots$. The spin connection is given by

$$\omega_{AB} = -\mathrm{d}e_A \cdot E_B + \mathrm{d}e_B \cdot E_A + \frac{1}{2}\left(e^C \cdot [E_A, E_B]\right) e_C . \tag{B.19}$$

Using the explicit expressions for E_m we have

$$[E_m, E_n] = [\Xi_m, \Xi_n] + \Sigma_{\hat{3}} \left(\Xi_m(A \cdot \Xi_n) - \Xi_n(A \cdot \Xi_m)\right) . \tag{B.20}$$

We can thus write, using the relation (A.12)

$$\omega_{mn} = \tilde{\omega}_{mn} + e^{\hat{3}} \rho_3 \frac{1}{2} \left(-A \cdot [\Xi_m, \Xi_n] + \Xi_m(A \cdot \Xi_n) - \Xi_n(A \cdot \Xi_m)\right) = \\ = \tilde{\omega}_{mn} + e^3 \rho_3 \Xi_m \cdot \mathrm{d}A \cdot \Xi_n \,. \tag{B.21}$$

In order to get the other components of the spin connection we will need the explicit form of the exterior derivative of $e^{\hat{a}} = \rho_{\hat{a}} \sigma^{\hat{a}} - \rho_3 \delta^{\hat{a}}_{\hat{3}} A_m e^m$ and of $e^{\tilde{a}} = \tilde{\rho} \sigma^{\tilde{a}}$

$$\mathrm{d}e^{\hat{a}} = \mathrm{d}\rho_{\hat{a}} \wedge \sigma^{\hat{a}} + \rho_{\hat{a}} \mathrm{d}\sigma^{\hat{a}} - \rho_3 \delta^{\hat{a}}_{\hat{3}} \mathrm{d}A - \delta^{\hat{a}}_{\hat{3}} \mathrm{d}\rho_3 \wedge A_m e^m \tag{B.22}$$

$$\mathrm{d}e^{\tilde{a}} = \mathrm{d}\rho_{\tilde{a}} \wedge \sigma^{\tilde{a}} + \rho_{\tilde{a}} \mathrm{d}\sigma^{\tilde{a}} \tag{B.23}$$

By definition

$$\begin{aligned} \mathrm{d}\sigma^{\hat{i}} &= \epsilon_{\hat{i}\hat{j}\hat{k}} \sigma^{\hat{j}} \wedge \sigma^{\hat{k}} \\ \mathrm{d}\sigma^{\tilde{i}} &= \epsilon_{\tilde{i}\tilde{j}\tilde{k}} \sigma^{\tilde{j}} \wedge \sigma^{\tilde{k}} \end{aligned} \tag{B.24}$$

and thus

$$\begin{aligned} [\Sigma_{\hat{a}}, \Sigma_{\hat{b}}] &= -2\epsilon_{\hat{c}\hat{a}\hat{b}} \Sigma_{\hat{c}} \\ [\Sigma_{\tilde{a}}, \Sigma_{\tilde{b}}] &= -2\epsilon_{\tilde{c}\tilde{a}\tilde{b}} \Sigma_{\tilde{c}} \,, \end{aligned} \tag{B.25}$$

so that

$$[E_{\hat{a}}, E_m] = \frac{1}{\rho_a^2} \partial_m \rho_a \Sigma_{\hat{a}} + \frac{1}{\rho_a} A_m [\Sigma_{\hat{a}}, \Sigma_{\hat{3}}] = \frac{1}{\rho_a^2} \partial_m \rho_a \Sigma_{\hat{a}} - \frac{2}{\rho_a} A_m \epsilon_{\hat{c}\hat{a}\hat{3}} \Sigma_{\hat{c}} \,. \tag{B.26}$$

In the end

$$\omega_{\hat{a}m} = -\mathrm{d}e_{\hat{a}} E_m + \frac{1}{2} e^P [E_{\hat{a}}, E_m] e_P = \\ = \partial_m \rho_a \sigma^{\hat{a}} + \delta^{\hat{a}}_{\hat{3}} e^p (\rho_3 \frac{1}{2} F_{pm} - A_p \partial_m \rho_3) \tag{B.27}$$

$$\omega_{\tilde{a}m} = -\mathrm{d}e_{\tilde{a}} E_m + \frac{1}{2} e^P [E_{\tilde{a}}, E_m] e_P = \partial_m \tilde{\rho} \, \sigma^{\tilde{a}} \tag{B.28}$$

and

$$\omega_{\hat{a}\hat{b}} = -\mathrm{d}e_{\hat{a}} E_{\hat{b}} + \mathrm{d}e_{\hat{b}} E_{\hat{a}} + \frac{1}{2} e^M \cdot [E_{\hat{a}}, E_{\hat{b}}] e_M = \\ = \epsilon_{\hat{a}\hat{b}\hat{c}} \left(\frac{\rho_a^2 + \rho_b^2 - \rho_c^2}{\rho_a \rho_b} \right) \sigma^{\hat{c}} + \epsilon_{\hat{a}\hat{b}\hat{3}} \frac{\rho_3^2}{\rho_1^2} A \tag{B.29}$$

$$\omega_{\tilde{a}\tilde{b}} = -\mathrm{d}e_{\tilde{a}} E_{\tilde{b}} + \mathrm{d}e_{\tilde{b}} E_{\tilde{a}} + \frac{1}{2} e^M \cdot [E_{\tilde{a}}, E_{\tilde{b}}] e_M = \epsilon_{\tilde{a}\tilde{b}\tilde{c}} \sigma^{\tilde{c}} \,. \tag{B.30}$$

The spin connection part of the covariant derivative acting on spinors as presented in appendix A is

$$\begin{aligned}
&\frac{1}{4}\omega_{MN}\Gamma^M\Gamma^N = \\
&\mathrm{d}x^\mu\Bigg[\frac{1}{4}\tilde{\omega}_{mn}\Gamma^m\Gamma^n - \frac{1}{4}\rho_3 F_{\mu m}\Gamma^m\Gamma^{\hat{3}} + \\
&\qquad - A_\mu\left(-\frac{1}{2}\frac{\rho_3^2}{\rho_1^2}\Gamma^{\hat{1}}\Gamma^{\hat{2}} - \frac{1}{2}\partial_m\rho_3\Gamma^m\Gamma^{\hat{3}} + \frac{1}{8}\rho_3^2 F_{mn}\Gamma^m\Gamma^n\right)\Bigg] + \\
&\sum_{a=1,2}\sigma^{\hat{a}}\left(\frac{1}{2}\frac{\rho_3}{\rho_1}\epsilon_{\hat{a}\hat{b}\hat{3}}\Gamma^{\hat{b}}\Gamma^{\hat{3}} - \frac{1}{2}\partial_m\rho_1\Gamma^m\Gamma^{\hat{a}}\right) + \\
&\quad + \sigma^{\hat{3}}\left(\frac{1}{2}\left(2 - \frac{\rho_3^2}{\rho_1^2}\right)\Gamma^{\hat{1}}\Gamma^{\hat{2}} - \frac{1}{2}\partial_m\rho_3\Gamma^m\Gamma^{\hat{3}} + \rho_3^2\frac{1}{8}F_{mn}\Gamma^m\Gamma^n\right) + \\
&\sum_{a=1,2,3}\sigma^{\tilde{a}}\left(\frac{1}{4}\epsilon_{\tilde{a}\tilde{b}\tilde{c}}\Gamma^{\tilde{b}}\Gamma^{\tilde{c}} - \frac{1}{2}\partial_m\tilde{\rho}\Gamma^m\Gamma^{\tilde{a}}\right)
\end{aligned} \tag{B.31}$$

B.3 Killing spinor

Conventions and Ansatz

We choose the following ten dimensional gamma matrices

$$\Gamma_m = \gamma_m \otimes 1 \otimes 1 \otimes 1 \quad \Gamma^{\hat{a}} = 1 \otimes \hat{\sigma}_1 \otimes \sigma_{\hat{a}} \otimes 1 \quad \Gamma^{\tilde{a}} = 1 \otimes \hat{\sigma}_2 \otimes 1 \otimes \sigma_{\tilde{a}} \tag{B.32}$$

The two 32 component Majorana-Weyl spinor supersymmetry parameters of the IIB theory can be grouped into a single complex Weyl spinor ψ obeying the chirality constraint

$$\Gamma_{11}\psi = \psi \tag{B.33}$$

$$\Gamma_{11} = \prod_m \Gamma_m \prod_{\hat{a}} \Gamma_{\hat{a}} \prod_{\tilde{a}} \Gamma_{\tilde{a}} = \gamma_5\hat{\sigma}_3 \qquad \gamma_5 = -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3\,. \tag{B.34}$$

The supersymmetry variation of the gravitino χ_M is given by

$$\delta\chi_M = \nabla_M\psi + \frac{\mathrm{i}}{480}F_{M_1M_2M_3M_4M_5}\Gamma^{M_1M_2M_3M_4M_5}\Gamma_M\psi\,. \tag{B.35}$$

In order to have a supersymmetric background we need to impose that this variation is zero giving rise to the Killing spinor equation on ψ,

$$\nabla_M\psi + \frac{\mathrm{i}}{480}F_{M_1M_2M_3M_4M_5}\Gamma^{M_1M_2M_3M_4M_5}\Gamma_M\psi = 0\,. \tag{B.36}$$

As a consequence of our symmetry assumptions we look for a ψ of the form

$$\psi = \varepsilon_{(b)} \otimes \hat{\chi} \otimes \tilde{\chi}_{(b)}\,. \tag{B.37}$$

Where ψ is an 8 component complex spinor a and $\hat{\chi}, \tilde{\chi}_b$ are 2 components complex spinors defined on the two 3-spheres satisfying

$$\Sigma_{\hat{a}}\hat{\chi} = 0 \qquad \sigma_{\hat{3}}\hat{\chi} = s\chi \tag{B.38}$$

$$\nabla'_{\tilde{a}}\tilde{\chi} = b\frac{\mathrm{i}}{2}\sigma_{\tilde{a}}\tilde{\chi}_{(b)} \tag{B.39}$$

where ∇' is the covariant derivative on the unit radius three sphere which has spin connection $\omega'_{abc} = \epsilon_{abc}$ and $s, b = \pm 1$. As we are going to show in the following, this choice means that $\hat{\chi}$ is a constant spinor and thus a singlet of the $SU(2)_L$ isometry of the squashed sphere, as required by our analysis of the gauge theory description of supersymmetries in section 2.

Isometries and Spinors

On a unit radius round three sphere there exist two linearly independent solutions to the equation

$$\nabla_a \chi = \beta \frac{\mathrm{i}}{2} \sigma_a \chi \tag{B.40}$$

for each choice of $\beta = \pm 1$. The sign of β is correlated with the chirality of the doublet of solutions under the $SO(4) = SU(2) \times SU(2)$ isometry group of S^3. This can be understood as follows.
Given a d-bein $e^a(y)$ and an isometry I we choose a local orthogonal transformation Λ such that

$$\Lambda^a{}_b \mathrm{T}I_*(e^b) = e^a \tag{B.41}$$

where $\mathrm{T}I_*$ is the pullback of one forms associated with I. The d-bein is thus invariant under these transformations and it is possible to give meaning to the transformation properties of spinors under the isometries of the metric.
In our case, since $S^3 \approx SU(2)$, we can identify the points y with elements of $SU(2)$. For the round 3-sphere $\tilde{S}^3$ the action of the isometry group $SU(2)_L \times SU(2)_R$ is given by left and right multiplication by generic elements of $SU(2)$. For the squashed three sphere the action of the isometry group $SU(2)_L \times U(1)_R$ is given by left multiplication by generic elements of $SU(2)$ and right multiplication with a $U(1)$ subgroup.
Let's focus on the left isometries L_g. They are defined by

$$L_g(y) = gy\,. \tag{B.42}$$

As our 3-bein is built out of left-invariant one forms σ^a, we have by definition

$$\mathrm{T}L_{g\,*}(\sigma^a) = \sigma^a \tag{B.43}$$

which implies that, for such transformations, $\Lambda^a{}_b = \delta^a{}_b$. The action $\mathrm{S}L_g$ on spinors of this isometry is thus very simple

$$\mathrm{S}L_g \chi(gy) = \chi(y)\,. \tag{B.44}$$

The action of left multiplications is clearly surjective and thus a spinor χ is invariant under this action if and only if it is a constant spinor. This means that our spinor $\hat{\chi}$ is a singlet under the $SU(2)_L$ isometry of the squashed 3-sphere, while the spinors $\tilde{\chi}_\pm$ transform in the $(0, \frac{1}{2})$ for upper sign and $(\frac{1}{2}, 0)$ for the lower sign. For a discussion of spinors in squashed 3-spheres see [37, 38].

Equations and bilinears

We turn now to the contribution of the Ramond-Ramond form to the gravitino variation. We define

$$M \equiv \frac{\mathrm{i}}{480} F_{M_1M_2M_3M_4M_5}\Gamma^{M_1M_2M_3M_4M_5}. \tag{B.45}$$

The chirality condition on ψ and the self-duality of $F_{(5)}$ imply that

$$M\Gamma_M\psi = -\left(\tilde{\not{G}} + \tilde{\not{V}}\gamma_5\hat{\sigma}_1\sigma_{\hat{3}} + \mathrm{i}\tilde{g}\sigma_{\hat{3}}\right)\gamma_5\hat{\sigma}_2\Gamma_M\psi\,. \tag{B.46}$$

Due to the conditions on the spinor, $\hat{\chi}$ and $\tilde{\chi}_b$ factorise in each component of the gravitino variation equation which then becomes the following system of coupled differential and algebraic equations on ε[7]

$$\left[\tilde{\nabla}_\mu - \frac{1}{4}F_{\mu\nu}\Xi^\nu{}_m\gamma^m\gamma^5\hat{\sigma}_1 s + \mathrm{i}A_\mu s - \left(\tilde{\not{G}} + \tilde{\not{V}}\gamma_5\hat{\sigma}_1 s + \mathrm{i}\tilde{g}s\right)\gamma_5\hat{\sigma}_2\gamma_\mu\right]\varepsilon = 0 \tag{B.48}$$

$$\left[\frac{\mathrm{i}}{2}\frac{\rho_3}{\rho_1}\gamma_5\hat{\sigma}_1 + \frac{1}{2}\tilde{\not{\partial}}\rho_1 + \rho_1\left(\tilde{\not{G}} + \tilde{\not{V}}\gamma_5\hat{\sigma}_1 s - \mathrm{i}\tilde{g}s\right)\gamma_5\hat{\sigma}_2\right]\varepsilon = 0 \tag{B.49}$$

$$\left[\frac{\mathrm{i}}{2}\left(2 - \frac{\rho_3^2}{\rho_1^2}\right)\gamma_5\hat{\sigma}_1 + \frac{1}{2}\tilde{\not{\partial}}\rho_3 + \frac{1}{8}\rho_3^2\not{F}\gamma_5\hat{\sigma}_1 s + \rho_3\left(\tilde{\not{G}} - \tilde{\not{V}}\gamma_5\hat{\sigma}_1 s + \mathrm{i}\tilde{g}s\right)\gamma_5\hat{\sigma}_2\right]\varepsilon = 0 \tag{B.50}$$

$$\left[\frac{\mathrm{i}}{2}b\gamma_5\hat{\sigma}_2 + \frac{1}{2}\tilde{\not{\partial}}\tilde{\rho} - \tilde{\rho}\left(\tilde{\not{G}} + \tilde{\not{V}}\gamma_5\hat{\sigma}_1 s + \mathrm{i}\tilde{g}s\right)\gamma_5\hat{\sigma}_2\right]\varepsilon = 0. \tag{B.51}$$

Note that the first equation is a first order differential 4-vector equation for ε while the last three are algebraic 4-scalar equations.
We now define a useful set of bilinears

$$\begin{aligned} &K_\mu = \bar{\varepsilon}\gamma_\mu\varepsilon \qquad L_\mu = \bar{\varepsilon}\gamma_5\gamma_\mu\varepsilon \qquad Y_{\mu\nu} = \bar{\varepsilon}\gamma_{\mu\nu}\sigma_1\varepsilon \\ &f_1 = \mathrm{i}\bar{\varepsilon}\sigma_1\varepsilon \qquad f_2 = \mathrm{i}\bar{\varepsilon}\sigma_2\varepsilon \\ &\bar{\varepsilon} = \varepsilon^\dagger\gamma_0 \end{aligned} \tag{B.52}$$

The world indices μ, ν of these bilinears are obtained by contraction of the tangent space indices with the vierbein $\varepsilon^m{}_\mu$. When raising and lowering μ indices we will always use the metric $\tilde{g}_{\mu\nu}$ unless otherwise is specified. By Fierz rearrangements the following relations can be proved

$$K^2 = -L^2 = -f_1^2 - f_2^2 \equiv -h^{-2} \qquad L^\mu K_\mu = 0 \tag{B.53}$$

[7]For example the first equation is obtained as follows

$$\begin{aligned} (\nabla_\mu + M\Gamma_\mu)\,\psi &= \left(\tilde{\nabla}_\mu - \frac{1}{4}\rho_3 F_{\mu\nu}\Xi^\nu{}_m\Gamma^m\Gamma^{\hat{3}} + A_\mu\left(\Sigma_{\hat{3}} + \Gamma^{\hat{1}}\Gamma^{\hat{2}}\right) - A_\mu\nabla_{\hat{3}} + M\Gamma_\mu\right)\psi = \\ &= \left(\tilde{\nabla}_\mu - \frac{1}{4}\rho_3 F_{\mu\nu}\Xi^\nu{}_m\Gamma^m\Gamma^{\hat{3}} + A_\mu\Gamma^{\hat{1}}\Gamma^{\hat{2}} + M\left(\Gamma_\mu + A_\mu\rho_3\Gamma_{\hat{3}}\right)\right)\psi = \\ &= \left(\tilde{\nabla}_\mu - \frac{1}{4}\rho_3 F_{\mu\nu}\Xi^\nu{}_m\gamma^m\sigma^{\hat{3}} + A_\mu\sigma_{\hat{3}} + M\gamma_\mu\right)\psi \end{aligned} \tag{B.47}$$

B.4 Algebraic relations

By multiplying the algebraic equations (B.49),(B.50),(B.51) with different combinations of gamma matrices and contracting with $\bar{\varepsilon}$ one can obtain the following relations for the spinor bilinears:

$$K^\mu \partial_\mu \rho_1 = 0 \tag{B.54}$$

$$K^\mu \partial_\mu \rho_3 = 0 \tag{B.55}$$

$$K^\mu \partial_\mu \tilde{\rho} = 0 \tag{B.56}$$

$$L_\mu = -\frac{\rho_1}{\rho_3}\frac{f_1}{\tilde{\rho}}\partial_\mu(\rho_1\tilde{\rho}) \tag{B.57}$$

$$K^\mu \tilde{V}_\mu = 0 \tag{B.58}$$

$$\tilde{g} = \frac{s}{4f_1}\left(b\frac{f_1}{\tilde{\rho}} - \frac{f_2\rho_3}{\rho_1^2}\right) \tag{B.59}$$

and also equations for the 2-forms $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\tilde{G}_{\mu\nu}$

$$\begin{aligned} F_{\mu\nu} = -\frac{2}{\rho_3(f_1^2+f_2^2)}\Big[&-\left(2-\frac{\rho_3^2}{\rho_1^2}\right)\frac{1}{\rho_3}\epsilon_{\mu\nu\rho\sigma}K^\rho L^\sigma + \frac{b}{\tilde{\rho}}\left(K_\mu L_\nu - K_\nu L_\mu\right) + \\ &- f_1\epsilon_{\mu\nu\rho\sigma}K^\rho\partial^\sigma\ln(\rho_3\tilde{\rho}) - f_2\big(K_\mu\partial_\nu\ln(\rho_3\tilde{\rho}) - K_\nu\partial_\mu\ln(\rho_3\tilde{\rho})\big) + \\ &+ 4f_1\big(K_\mu\tilde{V}_\nu - K_\nu\tilde{V}_\mu\big) + 4f_2\epsilon_{\mu\nu\rho\sigma}K^\sigma\tilde{V}^\rho\Big] \end{aligned} \tag{B.60}$$

$$\begin{aligned} \tilde{G}_{\mu\nu} = -\frac{1}{2(f_1^2+f_2^2)}\Big[&\left(\frac{b}{2\tilde{\rho}} - \tilde{g}s\right)\big(f_1(K_\mu\partial_\nu\ln\tilde{\rho} - K_\nu\partial_\mu\ln\tilde{\rho}) + f_2\epsilon_{\mu\nu\rho\sigma}K^\rho\partial^\sigma\ln\tilde{\rho}\big) + \\ &- f_2\left(K_\mu\tilde{V}_\nu - K_\nu\tilde{V}_\mu\right) + f_1\epsilon_{\mu\nu\rho\sigma}K^\rho\tilde{V}^\sigma\Big] \end{aligned} \tag{B.61}$$

B.5 Differential relations

We can use (B.48) to prove the following relations

$$\tilde{\nabla}_\mu K_\nu = 4\left(\tilde{G}_{\mu\nu}f_1 + G_{\mu\nu}f_2\right) - \frac{\rho_3}{2}F_{\mu\nu}f_2 s + 2\epsilon_{\mu\nu\rho\sigma}\tilde{V}^\rho K^\sigma s - 2\tilde{g}Y_{\mu\nu}s \tag{B.62}$$

$$\partial_\mu \ln f_1 = \partial_\mu \ln\tilde{\rho} \tag{B.63}$$

$$\partial_\mu\left(\frac{f_2}{\rho_3}\right) = F_{\mu\nu}K^\nu s\,. \tag{B.64}$$

The first equation says that $K^\mu\partial_\mu$ is a Killing vector for $g_{\mu\nu}$. We make the natural gauge choice

$$K^\mu\partial_\mu = \partial_t\,. \tag{B.65}$$

The second equation can be easily integrated to give, with a suitable choice of constant of integration

$$f_1 = \tilde{\rho}\,. \tag{B.66}$$

Note further that as a consequence of these equations and of the Bianchi identity

$$\tilde{V} = \frac{1}{2}\frac{1}{\rho_3\tilde{\rho}^3}\mathrm{d}(\tilde{g}\rho_1^2\tilde{\rho}^3)\,, \tag{B.67}$$

that $F_{\mu\nu}$ is t independent and we can make a gauge choice for A_μ such that $\partial_t A_\mu = 0$. Integrating the equation for f_2 we obtain

$$f_2 = \rho_3(c + A_t s)\,. \tag{B.68}$$

We define the coordinate

$$y \equiv \rho_1\tilde{\rho} \tag{B.69}$$

and thus

$$L_\mu \mathrm{d}x^\mu = -\frac{\rho_1}{\rho_3}\mathrm{d}y\,. \tag{B.70}$$

Since $K \cdot L = 0$ there is no cross term g_{ty} in the metric. We can additionally make a coordinate choice such that there are also no g_{yi} cross terms. We have thus reduced our Ansatz for the four dimensional part of the metric to the following

$$\mathrm{d}s^2 = -h^{-2}(\mathrm{d}t + V_1\mathrm{d}x^1 + V_2\mathrm{d}x^2)^2 + h^2\frac{\rho_1^2}{\rho_3^2}\tilde{h}_{ij}\mathrm{d}x^i\mathrm{d}x^j + h^2\frac{\rho_1^2}{\rho_3^2}\mathrm{d}y^2\,. \tag{B.71}$$

Note that

$$h^{-2} = f_1^2 + f_2^2\,. \tag{B.72}$$

For convenience we set

$$A_y = 0\,. \tag{B.73}$$

All the entries in the metric and in the 5-form are parametrised by a set of functions that we can distinguish on the basis of their transformation properties in the $\{x^1, x^2\}$ plane.

Scalars	Vectors	Symmetric Tensor
$\rho_1, \rho_3, \tilde{\rho}, A_t$	V_i, A_i	$\tilde{h}_{ij}$

Recalling that the scalars are subject to the constraint

$$y = \rho_1\tilde{\rho}\,. \tag{B.74}$$

¿From now on we will assume for definiteness that $s = 1$.

B.6 Specifying the spinor

Due to our gauge choice we have

$$K^0 = e^0{}_t = h^{-1} \quad \Rightarrow \quad \varepsilon^\dagger\varepsilon = h^{-1} \tag{B.75}$$

$$L_3 = L_y E^y{}_3 = -\frac{\rho_1}{\rho_3}\frac{\rho_3}{\rho_1}h^{-1} = -h^{-1}\,. \tag{B.76}$$

$$\tag{B.77}$$

We thus have:

$$\frac{\varepsilon^\dagger\gamma_0\gamma_5\gamma_3\varepsilon}{\varepsilon^\dagger\varepsilon} = -1 \quad \Rightarrow \quad \mathrm{i}\gamma_1\gamma_2\varepsilon = -\varepsilon\,. \tag{B.78}$$

We can now take the sum of equations (B.51) and (B.49) divided by, respectively, $\tilde{\rho}$ and ρ_1 from which we obtain

$$\big(\sqrt{1+\mathrm{e}^{-2G}}\gamma_3\hat{\sigma}_1 + \mathrm{i}\gamma_5\mathrm{e}^{-G} - 1\big)\varepsilon = 0 \tag{B.79}$$

where $\mathrm{e}^{-G} \equiv \frac{f_1}{f_2}$. The solution to this equation is given by

$$\varepsilon = \mathrm{e}^{\mathrm{i}\delta\gamma_5\gamma_3\hat{\sigma}_1}\varepsilon_1 \qquad \gamma_3\hat{\sigma}_1\varepsilon_1 = \varepsilon_1 \tag{B.80}$$

with $\sinh(2\delta) = \mathrm{e}^{-G}$. The normalization $h^{-1} = \varepsilon^\dagger\varepsilon$ implies $\varepsilon_1 = f_2^{1/2}\varepsilon_0$ with $\varepsilon_0^\dagger\varepsilon_0 = 1$. These conditions are enough to satisfy all the algebraic equations (B.49),(B.50),(B.51).

Due to the three projectors (B.33),(B.78),(B.80) and the conditions on the $\hat{\chi}, \tilde{\chi}$ spinors, the solution space of the Killing spinor equation is two dimensional and complex.

We will now use the differential equations (B.48) and the Bianchi identities (3.13)-(3.16) to express the unknown vectors and tensors in terms of the scalars.

B.7 The spacetime metric and the gauge field A

We define three new bilinears

$$\begin{aligned} \omega_\mu &= \varepsilon^t\gamma_2\gamma_\mu\varepsilon \\ W^{1,2}_{\mu\nu} &= \varepsilon^t\gamma_2\gamma_\mu\gamma_\nu\hat{\sigma}_{1,2}\varepsilon\,. \end{aligned} \tag{B.81}$$

Using (B.48) we can derive

$$\partial_\mu\omega_\nu - \partial_\nu\omega_\mu = -\mathrm{i}\frac{\rho_3}{2}F_\mu{}^\rho W^2_{\nu\rho} - 2\mathrm{i}(A_\mu\omega_\nu - A_\nu\omega_\mu) + 4\epsilon_{\mu\nu\rho\sigma}\tilde{V}^\rho\omega^\sigma - 4\tilde{g}W^1_{\mu\nu}\,. \tag{B.82}$$

We note that

$$\omega_\mu \mathrm{d}x^\mu = -\frac{\rho_1}{\rho_3}(\tilde{e}^1{}_j + \mathrm{i}\tilde{e}^2{}_j)\mathrm{d}x^j \equiv -\frac{\rho_1}{\rho_3}\tilde{e}^z{}_j\mathrm{d}x^j \tag{B.83}$$

where $\tilde{e}^k{}_j$ is a 2-bein for the metric $\tilde{h}_{ij}$. Thus, from (B.82) we can get an equation involving $\mathrm{d}\tilde{e}^k$. Singling out the y dependence using the (y, x^i) component of (B.82)

$$\partial_y\tilde{e}^z{}_j = -2\frac{h^2}{\rho_3\rho_1}\left[\frac{\tilde{\rho}}{\rho_3}(\rho_3^3 - \rho_1^2) + \frac{f_2}{\tilde{\rho}}(f_2\rho_3 - b\rho_1^2)\right]\tilde{e}^z{}_j \equiv D\tilde{e}^z{}_j\,. \tag{B.84}$$

With a further y independent coordinate transformation we can put $\tilde{h}_{ij}$ in diagonal form. We introduce a conformal factor T and set

$$\tilde{e}^i{}_j = T\delta^i{}_j \qquad \partial_y T = DT \tag{B.85}$$

Looking at the (x^1, x^2) component we can establish a relation between the remaining derivatives of T and the connection A_i

$$A_i = (A_t + b - c)V_i - \frac{1}{2}\epsilon_{ij}\partial_j\ln T\,. \tag{B.86}$$

The constant c can be absorbed into a gauge transformation and we will set

$$b = c = 1\,. \tag{B.87}$$

The right hand side of the $\{t, x^i\}$ component of equation (B.82) is proportional to $b - c$ and thus also this equation is consistent with our gauge choice.
We have now an expression for A_i by which we may calculate the components of $F_{\mu\nu}$. This $F_{\mu\nu}$ must be equal to the one obtained in (B.60). The contraction with K^μ is trivial. The F_{yi} components give the constraint

$$b = c \tag{B.88}$$

which is solved by our gauge choice. The F_{12} component gives an equation for $(\partial_1^2 + \partial_2^2)T$ that we will discuss later.

We have thus reduced our set of unknowns to five scalars and one 2-vector. Two scalars are constrained by the relations $y = \rho_1\tilde{\rho}$ and so we have just four independent scalars and one 2-vector.

Scalars	Vector
$\rho_1, \rho_3, \tilde{\rho}, A_t, T$	V_i

We have reduced the four dimensional metric to the form

$$\mathrm{d}s^2 = -h^{-2}(\mathrm{d}t + V_i\mathrm{d}x^i)^2 + h^2\frac{\rho_1^2}{\rho_3^2}(T^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + \mathrm{d}y^2)\,. \tag{B.89}$$

To simplify the final equations we now express the 4 functions $\rho_1, \rho_3, \tilde{\rho}, A_t$ in terms of three independent functions that we will call m, n, p are defined by

$$\begin{array}{ll} \rho_1^4 = y^4\frac{mp+n^2}{m} & \rho_3^4 = \frac{p^2}{m(mp+n^2)} \\ \tilde{\rho}^4 = \frac{m}{mp+n^2} & A_t = \frac{n-p}{p} \end{array} \tag{B.90}$$

With these definitions we have

$$D = 2y(n + m - y^{-2})\,. \tag{B.91}$$

With some effort it can be shown that all the equations on the spinor ε are now solved.

As noted in section B.6 the space of solutions to the Killing spinor equation is 2-dimensional and complex, thus our backgrounds preserve 4 of the 32 real supersymmetries of the theory. The existence of the Killing spinors guarantees that the full Einstein equations are satisfied provided that integrability conditions and the Bianchi identities for the Ramond-Ramond 5-form are satisfied. Let us now investigate what the consequence of these final constraints are.

B.8 Differential equations

We will first establish a relation between the vector V_i and the various scalar functions. The equation (B.62) is an equation for $\mathrm{d}K$ with

$$K = -h^{-2}(\mathrm{d}t + V_i\mathrm{d}x^i)\,. \tag{B.92}$$

We can extract from this equation an expression for $\mathrm{d}V$:

$$\mathrm{d}V = -y \star_3 [\mathrm{d}n + (nD + 2ym(n-p) + 2n/y)\mathrm{d}y] \tag{B.93}$$

where by $\star_3$ we mean the Hodge dual in the three dimensional diagonal metric

$$\mathrm{d}s_3^2 = T^2 \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j + \mathrm{d}y^2 \,. \tag{B.94}$$

Returning to the Bianchi identities

$$\mathrm{d}\big(\tilde{G}\tilde{\rho}^3 - \tilde{V} \wedge A\rho_3\tilde{\rho}^3\big) = 0 \tag{B.95}$$

$$\tilde{V} = \frac{1}{2}\frac{1}{\rho_3\tilde{\rho}^3}\mathrm{d}(\tilde{g}\rho_1^2\tilde{\rho}^3) \tag{B.96}$$

$$\mathrm{d}\big(G\rho_1^2\rho_3\big) = 0 \tag{B.97}$$

$$\mathrm{d}\big(G\rho_1^2\rho_3 \wedge A + \star_4\tilde{V}\big) - 2\star_4 \tilde{g} = 0 \,. \tag{B.98}$$

Substituting in the first equation $\tilde{V}$ as obtained from the second equation we find

$$\mathrm{d}\big(\tilde{G}\tilde{\rho}^3 - \frac{1}{2}\tilde{g}\rho_1^2\tilde{\rho}^3 F\big) = 0 \,. \tag{B.99}$$

We may thus set locally

$$\begin{aligned} \mathrm{d}\tilde{B} &= \tilde{G}\tilde{\rho}^3 - \frac{1}{2}\tilde{g}\rho_1^2\tilde{\rho}^3 F \\ \tilde{B} &= \tilde{B}_t(\mathrm{d}t + V) + \hat{\tilde{B}} \\ \mathrm{d}B &= G\rho_1^2\rho_3 \\ B &= B_t(\mathrm{d}t + V) + \hat{B} \,. \end{aligned} \tag{B.100}$$

The algebraic equation (B.61) for $\tilde{G}_{\mu\nu}$ and for its dual for $G_{\mu\nu}$ give rise to four new relations

$$\begin{aligned} \tilde{B}_t &= -\frac{1}{16}y^2\,\frac{n - 1/y^2}{p} \\ \mathrm{d}\hat{\tilde{B}} &= -\frac{1}{16}y^3 \star_3 [\mathrm{d}m + 2mD] \\ B_t &= -\frac{1}{16}y^2\,\frac{n}{m} \\ \mathrm{d}\hat{B} &= \frac{1}{16}y^3 \star_3 [\mathrm{d}p + 4yn(p-n)\mathrm{d}y] \,. \end{aligned} \tag{B.101}$$

We need to impose the three equations

$$\begin{cases} \mathrm{d}\mathrm{d}V = 0 \\ \mathrm{d}\mathrm{d}\hat{\tilde{B}} = 0 \\ \mathrm{d}\mathrm{d}\hat{B} = 0 \end{cases} . \tag{B.102}$$

The last Bianchi identity (B.98) is implied by these three. In addition to these equations we have also

$$\partial_y \ln T = D \tag{B.103}$$

which together with the previous ones can be used to see that also the consistency equation for F_{12} is satisfied.

We have thus a set of 4 equations for 4 unknowns: m, n, p, T. The equations are defined on the half space

$$(x^1, x^2, y > 0) \tag{B.104}$$

and are quite complicated being a set of coupled non-linear second order elliptic differential equations.

$$\begin{aligned}
&y^3(\partial_1^2+\partial_2^2)n + \partial_y\left(y^3T^2\partial_y n\right) + y^2\partial_y\big[T^2\big(yDn + 2y^2m(n-p)\big)\big] + 4y^2DT^2n = 0\\
&y^3(\partial_1^2+\partial_2^2)m + \partial_y\left(y^3T^2\partial_y m\right) + \partial_y\left(y^3T^2 2mD\right) = 0\\
&y^3(\partial_1^2+\partial_2^2)p + \partial_y\left(y^3T^2\partial_y p\right) + \partial_y\big[y^3T^2 4ny(n-p)\big] = 0
\end{aligned} \tag{B.105}$$

C. Killing vectors and the Kaluza Klein ansatz

In this appendix we present a geometrical interpretation of the bilinears that we constructed and that we used in appendix B

Assume we have a fibration of a group manifold over some d dimensional base manifold with metric

$$\mathrm{d}s^2 = \tilde{g}_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu + \beta_{ab}(x)\left(\hat{e}^a(y) - A^a_\mu(x)\mathrm{d}x^\mu\right)\left(\hat{e}^b(y) + A^b_\mu(x)\mathrm{d}x^\mu\right) \tag{C.1}$$

where $\hat{e}^a$ is a basis of left-invariant one forms on the group manifold.

We define

$$\kappa = K^\mu\partial_\mu + \alpha(x)^a\hat{E}_a\,. \tag{C.2}$$

We recall that given any covariant 2-tensor a and three vector W, V_1, V_2 the Lie derivative of a is given by

$$(\mathcal{L}_W a)(V_1, V_2) = W\left(a(V_1, V_2)\right) - a([W, V_1], V_2) - a(V_1, [W, V_2])\,. \tag{C.3}$$

Let us calculate $\mathcal{L}_K g$

$$\begin{gathered}
\left(\mathcal{L}_\kappa g\right)(\partial_\mu, \partial_\nu) = \left(\mathcal{L}_K\tilde{g}\right)(\partial_\mu, \partial_\nu) + K^\rho\partial_\rho\left(\beta_{ab}A^a_\mu A^b_\nu\right) - \partial_\mu\alpha^a\beta_{ab}A^b_\nu - \partial_\nu\alpha^a\beta_{ab}A^b_\mu\\
\left(\mathcal{L}_\kappa g\right)(\hat{E}_a, \hat{E}_b) = K^\rho\partial_\rho\beta_{ab} + \left(\mathcal{L}_\alpha\hat{g}(x)\right)(\hat{E}_a, \hat{E}_b)\\
\left(\mathcal{L}_\kappa g\right)(\partial_\mu, \hat{E}_a) = K^\rho\partial_\rho(-\beta_{ab}A^b_\mu) + \beta_{ab}\partial_\mu\alpha^b + \beta_{cd}\alpha^b A^d_\mu f^c_{ba}
\end{gathered}$$

where $\hat{g} = \beta_{ab}\hat{e}^a\hat{e}^b$ and f^c_{ba} the structure constants of the group.

When $K = 0, \beta_{ab} = k_{ab}$ with k_{ab} the Killing form of the group and so we obtain the non abelian Kaluza Klein setup.

Assume for the moment that K is a Killing vector of $\tilde{g}$ and $\alpha^a\hat{E}_a$ is a Killing vector of $\hat{g}$, what are the conditions on $\alpha, \beta_{ab}, A^a_\mu$ such that K is a Killing vector for the whole metric? This is easily seen from our previous equations

$$K(\beta_{ab}) = 0 \tag{C.4}$$

$$\partial_\mu\alpha^a = K(A^a_\mu) - f^a_{bc}\alpha^b A^c_\mu\,. \tag{C.5}$$

We can now specialise to our setting. The group manifold is $SU(2) \times SU(2)$. We can define the ten dimensional vector

$$\begin{aligned} &\kappa^M \partial_M = \bar{\psi}\Gamma^M \psi \partial_M = \\ &K^\mu \partial_\mu + \left(A_m K^m - \frac{f_2}{\rho_3} s \right) \Sigma^{\hat{i}}_{\hat{3}} \partial_{\hat{i}} + \frac{f_1}{\tilde{\rho}} \tilde{\chi}^\dagger \sigma^{\tilde{a}} \tilde{\chi} \Sigma^{\tilde{i}}_{\tilde{a}} \partial_{\tilde{i}} \end{aligned} \tag{C.6}$$

where we have chosen the normalization $\chi^\dagger \chi = \tilde{\chi}^\dagger \chi = 1$. κ is a Killing vector and it is null [39]. Since it is null we have

$$K^2 = K_\mu \tilde{g}^{\mu\nu} K^\nu = -f_1^2 - f_2^2 \tag{C.7}$$

which was previously seen as consequence of Fierz rearrangements and whereas here we can see its geometrical origin. From the equation on $\nabla_\mu K_\nu$ we know that K is a Killing vector for $\tilde{g}_{\mu\nu}$, and moreover, due to the Killing equation on $\tilde{\chi}$ and the properties of the Ansatz, we have that

$$\Sigma^{\hat{i}}_{\hat{3}} \qquad , \qquad \tilde{\chi}^\dagger \sigma^{\tilde{a}} \tilde{\chi} \Sigma^{\tilde{i}}_{\tilde{a}} \partial_{\tilde{i}} \tag{C.8}$$

are Killing vector of the group manifolds. We thus conclude

$$K(\rho_1) = K(\rho_3) = K(\tilde{\rho}) = 0 \tag{C.9}$$

$$\partial_\mu \left(\frac{f_1}{\tilde{\rho}} \right) = 0 \tag{C.10}$$

$$\partial_\mu \left(A_\nu K^\nu - \frac{f_2}{\rho_3} s \right) = K(A_\mu) \tag{C.11}$$

The second one can be written in the form we already encountered earlier

$$\partial_\mu \left(\frac{f_2}{\rho_3} \right) = F_{\mu\nu} K^\nu s \,. \tag{C.12}$$

We have thus clarified the geometrical origin of the relations between f_1, f_2 and the metric entries.

References

[1] H. Lin, O. Lunin and J.M. Maldacena, *Bubbling AdS space and 1/2 BPS geometries*, *JHEP* **10** (2004) 025 [hep-th/0409174].

[2] S. Corley, A. Jevicki and S. Ramgoolam, *Exact correlators of giant gravitons from dual $N = 4$ SYM theory*, *Adv. Theor. Math. Phys.* **5** (2002) 809 [hep-th/0111222].

[3] D. Berenstein, *A toy model for the AdS/CFT correspondence*, *JHEP* **07** (2004) 018 [hep-th/0403110].

[4] N.V. Suryanarayana, *Half-BPS giants, free fermions and microstates of superstars*, *JHEP* **01** (2006) 082 [hep-th/0411145].

[5] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the giant gravitons from anti-de Sitter space*, *JHEP* **06** (2000) 008 [hep-th/0003075].

[6] M.T. Grisaru, R.C. Myers and O. Tafjord, *SUSY and Goliath*, *JHEP* **08** (2000) 040 [hep-th/0008015].

[7] A. Hashimoto, S. Hirano and N. Itzhaki, *Large branes in AdS and their field theory dual*, *JHEP* **08** (2000) 051 [hep-th/0008016].

[8] J.T. Liu, D. Vaman and W.Y. Wen, *Bubbling 1/4 BPS solutions in type-IIB and supergravity reductions on $S^n \times S^n$*, *Nucl. Phys.* **B 739** (2006) 285 [hep-th/0412043].

[9] Z.W. Chong, H. Lu and C.N. Pope, *BPS geometries and AdS bubbles*, *Phys. Lett.* **B 614** (2005) 96 [hep-th/0412221].

[10] A. Donos, *A description of 1/4 BPS configurations in minimal type-IIB SUGRA*, *Phys. Rev.* **D 75** (2007) 025010 [hep-th/0606199].

[11] A. Donos, *BPS states in type-IIB sugra with $SO(4) \times SO(2)$(gauged) symmetry*, hep-th/0610259.

[12] D. Berenstein, *Large-N BPS states and emergent quantum gravity*, *JHEP* **01** (2006) 125 [hep-th/0507203].

[13] A. Mikhailov, *Giant gravitons from holomorphic surfaces*, *JHEP* **11** (2000) 027 [hep-th/0010206].

[14] G. Mandal and N.V. Suryanarayana, *Counting 1/8-BPS dual-giants*, *JHEP* **03** (2007) 031 [hep-th/0606088].

[15] I. Biswas, D. Gaiotto, S. Lahiri and S. Minwalla, *Supersymmetric states of $N = 4$ Yang-Mills from giant gravitons*, hep-th/0606087.

[16] N. Kim, *AdS_3 solutions of IIB supergravity from D3-branes*, *JHEP* **01** (2006) 094 [hep-th/0511029].

[17] N. Kim and J.-D. Park, *Comments on AdS_2 solutions of $D = 11$ supergravity*, *JHEP* **09** (2006) 041 [hep-th/0607093].

[18] O. Lunin, *On gravitational description of Wilson lines*, *JHEP* **06** (2006) 026 [hep-th/0604133].

[19] J.P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, *G-structures and wrapped NS5-branes*, *Commun. Math. Phys.* **247** (2004) 421 [hep-th/0205050].

[20] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, *All supersymmetric solutions of minimal supergravity in five dimensions*, *Class. and Quant. Grav.* **20** (2003) 4587 [hep-th/0209114].

[21] J.B. Gutowski, D. Martelli and H.S. Reall, *All supersymmetric solutions of minimal supergravity in six dimensions*, *Class. and Quant. Grav.* **20** (2003) 5049 [hep-th/0306235].

[22] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Supersymmetric AdS_5 solutions of M-theory*, *Class. and Quant. Grav.* **21** (2004) 4335 [hep-th/0402153].

[23] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Sasaki-Einstein metrics on $S(2) \times S(3)$*, *Adv. Theor. Math. Phys.* **8** (2004) 711 [hep-th/0403002].

[24] V.K. Dobrev and V.B. Petkova, *All positive energy unitary irreducible representations of extended conformal supersymmetry*, *Phys. Lett.* **B 162** (1985) 127.

JHEP05(2007)030

[25] L. Andrianopoli and S. Ferrara, *Short and long su(2,2/4) multiplets in the AdS/CFT correspondence*, *Lett. Math. Phys.* **48** (1999) 145 [hep-th/9812067].

[26] L. Andrianopoli, S. Ferrara, E. Sokatchev and B. Zupnik, *Shortening of primary operators in n-extended sCFT(4) and harmonic-superspace analyticity*, *Adv. Theor. Math. Phys.* **4** (2000) 1149 [hep-th/9912007].

[27] A.V. Ryzhov, *Quarter BPS operators in $N = 4$ SYM*, *JHEP* **11** (2001) 046 [hep-th/0109064].

[28] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[29] G. Milanesi and M. O'Loughlin, *Singularities and closed time-like curves in type-IIB 1/2 BPS geometries*, *JHEP* **09** (2005) 008 [hep-th/0507056].

[30] E. Gava, G. Milanesi, K. Narain, and M. O'Loughlin, in preparation.

[31] L. Grant, L. Maoz, J. Marsano, K. Papadodimas and V.S. Rychkov, *Minisuperspace quantization of 'bubbling AdS' and free fermion droplets*, *JHEP* **08** (2005) 025 [hep-th/0505079].

[32] L. Maoz and V.S. Rychkov, *Geometry quantization from supergravity: the case of 'bubbling AdS'*, *JHEP* **08** (2005) 096 [hep-th/0508059].

[33] R.C. Myers and O. Tafjord, *Superstars and giant gravitons*, *JHEP* **11** (2001) 009 [hep-th/0109127].

[34] M.M. Caldarelli, D. Klemm and P.J. Silva, *Chronology protection in anti-de Sitter*, *Class. and Quant. Grav.* **22** (2005) 3461 [hep-th/0411203].

[35] V. Balasubramanian, J. de Boer, V. Jejjala and J. Simon, *The library of babel: on the origin of gravitational thermodynamics*, *JHEP* **12** (2005) 006 [hep-th/0508023].

[36] E. Gava, G. Milanesi, K. Narain, and M. O'Loughlin, in preparation.

[37] G.W. Gibbons, *Spectral asymmetry and quantum field theory in curved space-time*, *Ann. Phys. (NY)* **125** (1980) 98.

[38] N. Hitchin, *Harmonic Spinors*, *Adv. Math* **14** (1974) 1.

[39] E.J. Hackett-Jones and D.J. Smith, *Type IIB killing spinors and calibrations*, *JHEP* **11** (2004) 029 [hep-th/0405098].

[40] K. Skenderis and M. Taylor, *Kaluza-Klein holography*, *JHEP* **05** (2006) 057 [hep-th/0603016].

JHEP05(2007)030